Stochastic processes: basic notions *

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1. Fundamental concepts

1.1. Probability space

Definition 1.1 PROBABLITY SPACE. *A* probability space *is a triplet* $(\Omega, \mathscr{A}, \mathbb{P})$ *where*

(1) Ω *is the set of all possible results of an experiment*;

(2) $\mathscr{A}$ *is a class of subsets of* Ω *(called* events*) forming a* $\sigma-$*algebra,* i.e.

(i) $\Omega \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $\bigcup_{j=1}^{\infty} A_j \in \mathscr{A}$ *, for any sequence* $\{A_1, A_2, ...\} \subseteq \mathscr{A}$;

(3) $\mathbb{P} : \mathscr{A} \to [0,1]$ *is a function which assigns to each event* $A \in \mathscr{A}$ *a number* $\mathbb{P}(A) \in [0,1]$, *called the probability of* A *and such that*

(i) $\mathbb{P}(\Omega) = 1$,

(ii) *if* $\{A_j\}_{j=1}^{\infty}$ *is a sequence of disjoint events, then* $\mathbb{P}(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$.

1.2. Real random variable

Definition 1.2 REAL RANDOM VARIABLE (HEURISTIC DEFINITION). *A* real random variable X *is a variable with real values whose behavior can be described by a probability distribution. Usually, this probability distribution is described by a distribution function:*

$$F_X(x) = \mathbb{P}[X \leq x] \ . \tag{1.1}$$

Definition 1.3 REAL RANDOM VARIABLE. *A* real random variable X *is a function* $X : \Omega \to \mathbb{R}$ *such that*

$$X^{-1}\big((-\infty, x]\big) := \{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{A}, \forall x \in \mathbb{R} \, .$$

X *is a* measurable function. *The probability distribution of* X *is defined by*

$$F_X(x) = \mathbb{P}[X^{-1}((-\infty, x])] \, . \tag{1.2}$$

1.3. Stochastic process

Definition 1.4 REAL STOCHASTIC PROCESS. *Let $\mathbb{T}$ be a non-empty set. A stochastic process on $\mathbb{T}$ is a collection of random variables $X_t : \Omega \to \mathbb{R}$ such that a random variable X_t is associated with each each element $t \in \mathbb{T}$. This stochastic process is denoted by $\{X_t : t \in \mathbb{T}\}$, or more simply by X_t when the definition of $\mathbb{T}$ is clear. If $\mathbb{T} = \mathbb{R}$ (real numbers), $\{X_t : t \in \mathbb{T}\}$ is a continuous time process. If $\mathbb{T} = \mathbb{Z}$ (integers) or $\mathbb{T} \subseteq \mathbb{Z}$, $X_t : t \in \mathbb{T}\}$ is discrete time process.*

The set $\mathbb{T}$ can be finite or infinite, though usually it is more convenient to assume that $\mathbb{T}$ is infinite. In the sequel, we focus on processes where $\mathbb{T}$ is a right-infinite interval of integers, *i.e.*

$$\mathbb{T} = (n_0, \infty) \text{ where } n_0 \in \mathbb{Z} \text{ or } n_0 = -\infty. \tag{1.3}$$

We can also consider random variables which take their values in more general spaces, *i.e.*

$$X_t : \Omega \to \Omega_0 \tag{1.4}$$

where Ω_0 is any non-empty set. Unless stated otherwise, we shall limit ourselves to the case where $\Omega_0 = \mathbb{R}$.

To observe a time series is equivalent to observing a realization of a process $\{X_t : t \in \mathbb{T}\}$ or a portion of such a realization: given $(\Omega, \mathscr{A}, \mathbb{P})$, $\omega \in \Omega$ is drawn first, and then the variables $X_t(\omega)$, $t \in \mathbb{T}$, are associated with it. Each realization is determined in one shot by ω.

The probability law of a stochastic process $\{X_t : t \in \mathbb{T}\}$ with $\mathbb{T} \subseteq \mathbb{R}$ can be described by specifying the joint distribution function of $(X_{t_1}, \ldots, X_{t_n})$ for each subset $\{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{T}$ (where $n \geq 1$):

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = \mathbb{P}[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] \,. \tag{1.5}$$

This follows from Kolmogorov's theorem [see Brockwell and Davis (1991, Chapter 1)].

1.4. L_r spaces

Definition 1.5 L_r SPACE. *Let r be a real number. L_r is the set of real random variables X defined on $(\Omega, \mathscr{A}, \mathbb{P})$ such that $\mathbb{E}[|X|^r] < \infty$.*

The space L_r is always defined with respect to a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. In other words, we have:

$$L_r = L_r(\Omega, \mathscr{A}, \mathbb{P}) . \tag{1.6}$$

L_2 is the set of random variables on $(\Omega, \mathscr{A}, \mathbb{P})$ whose second moments are finite (*square-integrable variables*). A stochastic process $\{X_t : t \in \mathbb{T}\}$ is in L_r iff $X_t \in L_r$, $\forall t \in \mathbb{T}$, *i.e.*

$$\mathbb{E}[|X_t|^r] < \infty \, , \forall t \in \mathbb{T} \, . \tag{1.7}$$

The properties of moments of random variables are summarized in Dufour (2022).

2. Stationary processes

In general, the variables of a process $\{X_t : t \in \mathbb{T}\}$ are not identically distributed nor independent. In particular, if we suppose that $\mathbb{E}(X_t^2) < \infty$, we have:

$$\mathbb{E}(X_t) = \mu_t \ , \tag{2.1}$$

$$\mathrm{Cov}(X_{t_1}, X_{t_2}) = \mathbb{E}[(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})] = C(t_1, t_2) \ . \tag{2.2}$$

The means, variances and covariances of the variables of the process depend on their position in the series. The behavior of X_t can change with time. The function $C : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is called the *covariance function* of the process $\{X_t : t \in \mathbb{T}\}$.

In this section, we will focus on the case where $\mathbb{T}$ is a right-infinite interval of integers.

Assumption 2.1 PROCESS ON AN INTERVAL OF INTEGERS.

$$\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\} \quad \textit{where } n_0 \in \mathbb{Z} \cup \{-\infty\} \, . \tag{2.3}$$

Definition 2.1 STRICTLY STATIONARY PROCESS. *A stochastic process $\{X_t : t \in \mathbb{T}\}$ is strictly stationary (SS) iff the probability distribution of the vector $(X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k})'$ is identical with the one of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})'$, for any finite subset $\{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{T}$ and any integer $k \geq 0$. To indicate that $\{X_t : t \in \mathbb{T}\}$ is SS, we write $\{X_t : t \in \mathbb{T}\} \sim SS$ or $X_t \sim SS$.*

Proposition 2.1 CHARACTERIZATION OF STRICT STATIONARITY FOR A PROCESS ON (n_0, ∞). *If the process $\{X_t : t \in \mathbb{T}\}$ is SS, then the probability distribution of the vector $(X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k})'$ is identical to the one of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})'$, for any finite subset $\{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{T}$ and any integer $k > n_0 - \min\{t_1, \ldots, t_n\}$.*

For processes on the integers ($\mathbb{T} = \mathbb{Z}$), the above characterization can be formulated in a simpler way as follows.

Proposition 2.2 CHARACTERIZATION OF STRICT STATIONARITY FOR A PROCESS ON THE INTEGERS. *A process $\{X_t : t \in \mathbb{Z}\}$ is SS iff the probability distribution of $(X_{t_1+k},\ X_{t_2+k}, \ldots, X_{t_n+k})'$ is identical with the probability distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})'$, for any subset $\{t_1,\ t_2, \ldots, t_n\} \subseteq \mathbb{Z}$ and any integer k.*

Definition 2.2 SECOND-ORDER STATIONARY PROCESS. *A stochastic process $\{X_t : t \in \mathbb{T}\}$ is* second-order stationary *(S2) iff*

$$\begin{aligned} &(1) \qquad \mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \\ &(2) \qquad \mathbb{E}(X_s) = \mathbb{E}(X_t), \forall s, t \in \mathbb{T}, \\ &(3) \qquad \mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+k}, X_{t+k}), \forall s, t \in \mathbb{T}, \forall k \geq 0\ . \end{aligned} \tag{2.4}$$

If $\{X_t : t \in \mathbb{T}\}$ is S2, we write $\{X_t : t \in \mathbb{T}\} \sim S2$ or $X_t \sim S2$.

Remark 2.1 Instead of *second-order stationary*, we also say *weakly stationary* (WS).

Proposition 2.3 RELATION BETWEEN STRICT AND SECOND-ORDER STATIONARITY. *If the process $\{X_t : t \in \mathbb{T}\}$ is strictly stationary and $\mathbb{E}(X_t^2) < \infty$ for any $t \in \mathbb{T}$, then the process $\{X_t : t \in \mathbb{T}\}$ is second-order stationary.*

PROOF. Suppose $\mathbb{E}(X_t^2) < \infty$, for any $t \in \mathbb{T}$. If the process $\{X_t : t \in \mathbb{T}\}$ is SS, we have:

$$\mathbb{E}(X_s) = \mathbb{E}(X_t) \ , \forall s, t \in \mathbb{T} \ , \tag{2.5}$$

$$\mathbb{E}(X_s X_t) = \mathbb{E}(X_{s+k} X_{t+k}) \ , \forall s, t \in \mathbb{T}, \forall k \geq 0 \ . \tag{2.6}$$

Since

$$\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_s X_t) - \mathbb{E}(X_s)\mathbb{E}(X_t) \ , \tag{2.7}$$

we see that

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+k}, X_{t+k}) \ , \forall s, t \in \mathbb{T} \ , \forall k \geq 0 \ , \tag{2.8}$$

so the conditions (2.5) - (2.8) are equivalent to the conditions (2.5) - (2.6). The mean of X_t is constant, and the covariance between any two variables of the process only depends on the distance between the variables, not their position in the series. □

Proposition 2.4 EXISTENCE OF AN AUTOCOVARIANCE FUNCTION. *If the process $\{X_t : t \in \mathbb{T}\}$ is second-order stationary, then there exists a function $\gamma : \mathbb{Z} \to \mathbb{R}$ such that*

$$\mathrm{Cov}(X_s, X_t) = \gamma(t-s)\ , \forall s, t \in \mathbb{T}. \tag{2.9}$$

The function γ is called the autocovariance function of the process $\{X_t : t \in \mathbb{T}\}$, and $\gamma_k := \gamma(k)$ the lag-k autocovariance of the process $\{X_t : t \in \mathbb{T}\}$.

PROOF. To show the existence of the autocovariance function in (2.9), we need to prove the following implication:

$$t_2 - s_2 = t_1 - s_1 \ \Rightarrow\ \mathrm{Cov}(X_{s_2}, X_{t_2}) = \mathrm{Cov}(X_{s_1}, X_{t_1}) \tag{2.10}$$

for all pairs (s_1, t_1) and (s_2, t_2). Suppose that $t_2 - s_2 = t_1 - s_1$. Then, using the stationarity assumption,

$$\begin{aligned}
\mathrm{Cov}(X_{s_2}, X_{t_2}) &= \mathrm{Cov}(X_{s_2+(s_1-s_2)}, X_{t_2+(s_1-s_2)}) \\
&= \mathrm{Cov}(X_{s_1}, X_{s_1+(t_2-s_2)}) \\
&= \mathrm{Cov}(X_{s_1}, X_{s_1+(t_1-s_1)}) \\
&= \mathrm{Cov}(X_{s_1}, X_{t_1})\,.
\end{aligned} \tag{2.11}$$

□

Proposition 2.5 PROPERTIES OF THE AUTOCOVARIANCE FUNCTION. *Let $\{X_t : t \in \mathbb{T}\}$ be a second-order stationary process. The autocovariance function $\gamma(k)$ of the process $\{X_t : t \in \mathbb{T}\}$ satisfies the following properties:*

(1) $\gamma(0) = V(X_t) \geq 0$, $\forall t \in \mathbb{T}$;

(2) $\gamma(k) = \gamma(-k)$, $\forall k \in \mathbb{Z}$ (i.e., $\gamma(k)$ *is an even function of* k);

(3) $|\gamma(k)| \leq \gamma(0)$, $\forall k \in \mathbb{Z}$;

(4) *the function $\gamma(k)$ is positive semidefinite,* i.e.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \gamma(t_i - t_j) \geq 0, \tag{2.12}$$

for any positive integer N and for all the vectors $a = (a_1, \ldots, a_N)' \in \mathbb{R}^N$ and $(t_1, \ldots, t_N)' \in \mathbb{T}^N$;

(5) *any $N \times N$ matrix of the form*

$$\begin{aligned} \Gamma_N &= [\gamma(j-i)]_{i,\ j=1,\ldots,N} \\ &= \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(N-1) \\ \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(N-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma(N-1) & \gamma(N-2) & \gamma(N-3) & \cdots & \gamma(0) \end{bmatrix} \end{aligned} \tag{2.13}$$

is positive semidefinite.

Proposition 2.6 EXISTENCE OF AN AUTOCORRELATION FUNCTION. *If the process* $\{X_t : t \in \mathbb{T}\}$ *is second-order stationary, then there exists a function* $\rho : \mathbb{Z} \to [-1, 1]$ *such that*

$$\rho(t-s) = \mathrm{Corr}(X_s, X_t) = \gamma(t-s)/\gamma(0)\ , \forall s, t \in \mathbb{T}\,, \tag{2.14}$$

where $0/0 := 1$. *The function* ρ *is called the* autocorrelation function *of the process* $\{X_t : t \in \mathbb{T}\}$, *and* $\rho_k := \rho(k)$ *the* lag-k autocorrelation *of the process* $\{X_t : t \in \mathbb{T}\}$.

Proposition 2.7 PROPERTIES OF THE AUTOCORRELATION FUNCTION. *Let $\{X_t : t \in \mathbb{T}\}$ be a second-order stationary process. The autocorrelation function $\rho(k)$ of the process $\{X_t : t \in \mathbb{T}\}$ satisfies the following properties:*

(1) $\rho(0) = 1$;

(2) $\rho(k) = \rho(-k)$, $\forall k \in \mathbb{Z}$;

(3) $|\rho(k)| \leq 1, \ \forall k \in \mathbb{Z}$;

(4) *the function $\rho(k)$ is positive semidefinite,* i.e.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \rho(t_i - t_j) \geq 0 \tag{2.15}$$

for any positive integer N and for all the vectors $a = (a_1, \ldots, a_N)' \in \mathbb{R}^N$ and $(t_1, \ldots, t_N)' \in \mathbb{T}^N$;

(5) *any $N \times N$ matrix of the form*

$$R_N = \frac{1}{\gamma_0}\Gamma_N = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \cdots & \rho(N-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(N-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(N-1) & \rho(N-2) & \rho(N-3) & \cdots & 1 \end{bmatrix} \tag{2.16}$$

is positive semidefinite, where $\gamma(0) = V(X_t)$.

Theorem 2.8 CHARACTERIZATION OF AUTOCOVARIANCE FUNCTIONS. *An even function $\gamma : \mathbb{Z} \to \mathbb{R}$ is positive semidefinite iff $\gamma(\cdot)$ is the autocovariance function of a second-order stationary process $\{X_t : t \in \mathbb{Z}\}$.*

PROOF. See Brockwell and Davis (1991, Chapter 2). □

Corollary 2.9 CHARACTERIZATION OF AUTOCORRELATION FUNCTIONS. *An even function $\rho : \mathbb{Z} \to [-1, 1]$ is positive semidefinite iff ρ is the autocorrelation function of a second-order stationary process $\{X_t : t \in \mathbb{Z}\}$.*

Definition 2.3 DETERMINISTIC PROCESS. *Let $\{X_t : t \in \mathbb{T}\}$ be a stochastic process, $\mathbb{T}_1 \subseteq \mathbb{T}$ and $I_t = \{X_s : s \leq t\}$. We say that the process $\{X_t : t \in \mathbb{T}\}$ is* deterministic *on $\mathbb{T}_1$ iff there exists a collection of functions $\{g_t(I_{t-1}) : t \in \mathbb{T}_1\}$ such that $X_t = g_t(I_{t-1})$ with probability one, $\forall t \in \mathbb{T}_1$.*

A deterministic process can be perfectly predicted form its own past (at points where it is deterministic).

Proposition 2.10 CRITERION FOR A DETERMINISTIC PROCESS. *Let $\{X_t : t \in \mathbb{T}\}$ be a second-order stationary process, where $\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\}$ and $n_0 \in \mathbb{Z} \cup \{-\infty\}$, and let $\gamma(k)$ its autocovariance function. If there exists an integer $N \geq 1$ such that the matrix Γ_N is singular [where Γ_N is defined in Proposition 2.5], then the process $\{X_t : t \in \mathbb{T}\}$ is deterministic for $t > n_0 + N - 1$. In particular, if $V(X_t) = \gamma(0) = 0$, the process is deterministic for $t \in \mathbb{T}$.*

For a second-order indeterministic stationary process at any $t \in \mathbb{T}$, all the matrices Γ_N, $N \geq 1$, are invertible.

Definition 2.4 STATIONARITY OF ORDER m. *Let m be a non-negative integer. A stochastic process $\{X_t : t \in \mathbb{T}\}$ is stationary of order m iff*

(1) $\mathbb{E}(|X_t|^m) < \infty, \forall t \in \mathbb{T}$,
and

(2) $\mathbb{E}[X_{t_1}^{m_1} X_{t_2}^{m_2} \cdots X_{t_n}^{m_n}] = \mathbb{E}[X_{t_1+k}^{m_1} X_{t_2+k}^{m_2} \cdots X_{t_n+k}^{m_n}]$
for any $k \geq 0$, and for any nonempty subset $\{t_1, \ldots, t_n\} \in \mathbb{T}^N$ and all the non-negative integers $m_1, \ldots, m_n$ such that $m_1 + m_2 + \cdots + m_n \leq m$.

If $m = 1$, the mean is constant, but not necessarily the other moments. If $m = 2$, the process is second-order stationary.

Definition 2.5 ASYMPTOTIC STATIONARITY OF ORDER m. *Let m a non-negative integer. A stochastic process $\{X_t : t \in \mathbb{T}\}$ is asymptotically stationary of order m iff*

(1) *there exists an integer N such that $(|X_t|^m) < \infty$, for $t \geq N$,*
and

(2) $\lim_{t_1 \to \infty} \left[\mathbb{E}\left(X_{t_1}^{m_1} X_{t_1+\Delta_2}^{m_2} \cdots X_{t_1+\Delta_n}^{m_n} \right) - \mathbb{E}\left(X_{t_1+k}^{m_1} X_{t_1+\Delta_2+k}^{m_2} \cdots X_{t_1+\Delta_n+k}^{m_n} \right) \right] = 0$
for any $k \geq 0$, $t_1 \in \mathbb{T}$, all the positive integers $\Delta_2, \Delta_3, \ldots, \Delta_n$ such that $\Delta_2 < \Delta_3 < \cdots < \Delta_n$, and all non-negative integers $m_1, \ldots, m_n$ such that $m_1 + m_2 + \cdots + m_n \leq m$.

3. Some important models

In this section, we will again assume that $\mathbb{T}$ is a right-infinite interval of integers (Assumption 2.1):

$$\mathbb{T} = \{t \in \mathbb{Z} : t > n_0\} \quad \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}\,. \tag{3.1}$$

3.1. Noise models

Definition 3.1 SEQUENCE OF INDEPENDENT RANDOM VARIABLES. *A process $\{X_t : t \in \mathbb{T}\}$ is a* sequence of independent *random variables iff the variables X_t are mutually independent. This is denoted by:*

$$\{X_t : t \in \mathbb{T}\} \sim IND \text{ or } \{X_t\} \sim IND\,. \tag{3.2}$$

Further, we write:

$$\{X_t : t \in \mathbb{T}\} \sim IND(\mu_t) \quad \text{if } \mathbb{E}(X_t) = \mu_t\,, \tag{3.3}$$

$$\{X_t : t \in \mathbb{T}\} \sim IND(\mu_t, \sigma_t^2) \quad \text{if } \mathbb{E}(X_t) = \mu_t \text{ and } \mathrm{V}(X_t) = \sigma_t^2\,.$$

Definition 3.2 RANDOM SAMPLE. *A* random sample *is a sequence of independent and identically distributed (*i.i.d.*) random variables. This is denoted by*

$$\{X_t : t \in \mathbb{T}\} \sim IID\,. \tag{3.4}$$

A random sample is a SS process. If $\mathbb{E}(X_t^2) < \infty$, for all $t \in \mathbb{T}$, the process is S2. In this case, we write:

$$\{X_t : t \in \mathbb{T}\} \sim IID(\mu, \sigma^2)\,, \quad \text{if } \mathbb{E}(X_t) = \mu \text{ and } \mathrm{V}(X_t) = \sigma^2. \tag{3.5}$$

Definition 3.3 WHITE NOISE. *A* white noise *is a sequence of random variables in L_2 with mean zero, the same variance and mutually uncorrelated,* i.e.

$$\mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \tag{3.6}$$

$$\mathbb{E}(X_t^2) = \sigma^2 \ , \forall t \in \mathbb{T}, \tag{3.7}$$

$$\mathrm{Cov}(X_s, X_t) = 0 \,, \ \textit{if } s \neq t \,. \tag{3.8}$$

This is denoted by:

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{WN}(0, \sigma^2) \textit{ or } \{X_t\} \sim \mathrm{WN}(0, \sigma^2) \,. \tag{3.9}$$

Definition 3.4 HETEROSKEDASTIC WHITE NOISE. *A* heteroskedastic white noise *is a sequence of random variables in L_2 with mean zero and mutually uncorrelated,* i.e.

$$\mathbb{E}(X_t^2) < \infty, \forall t \in \mathbb{T}, \tag{3.10}$$

$$\mathbb{E}(X_t) = 0, \forall t \in \mathbb{T}, \tag{3.11}$$

$$\mathrm{Cov}(X_t, X_s) = 0, \ \textit{if } s \neq t \,, \tag{3.12}$$

$$\mathbb{E}(X_t^2) = \sigma_t^2 \,, \ \forall t \in \mathbb{T} \,. \tag{3.13}$$

This is denoted by:

$$\{X_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma_t^2) \textit{ or } \{X_t\} \sim \mathrm{WN}(0, \sigma_t^2) \,. \tag{3.14}$$

Each one of these four models will be called a *noise* process.

3.2. Harmonic processes

Many time series exhibit apparent periodic behavior. This suggests one to use periodic functions to describe them.

Definition 3.5 PERIODIC FUNCTION. *A function $f(t), t \in \mathbb{R}$, is periodic of period P on $\mathbb{R}$ iff*

$$f(t+P) = f(t)\,,\ \forall t, \tag{3.15}$$

and P is the lowest number such that (3.15) holds for all t. $\frac{1}{P}$ is the frequency associated with the function (number of cycles per unit of time).

Example 3.1 Sinus function:

$$\sin(t) = \sin(t+2\pi) = \sin(t+2\pi k), \forall k \in \mathbb{Z}. \tag{3.16}$$

For the sinus function, the period is $P = 2\pi$ and the frequency is $f = 1/(2\pi)$.

Example 3.2 Cosine function:

$$\cos(t) = \cos(t+2\pi) = \cos(t+2\pi k), \forall k \in \mathbb{Z}. \tag{3.17}$$

Example 3.3

$$\sin(\nu t) = \sin\left[\nu\left(t+\frac{2\pi}{\nu}\right)\right] = \sin\left[\nu\left(t+\frac{2\pi k}{\nu}\right)\right]\ , \forall k \in \mathbb{Z}. \tag{3.18}$$

Example 3.4

$$\cos(\nu t) = \cos\left[\nu\left(t + \frac{2\pi}{\nu}\right)\right] = \cos\left[\nu\left(t + \frac{2\pi k}{\nu}\right)\right] \ , \forall k \in \mathbb{Z}. \tag{3.19}$$

For $\sin(\nu t)$ and $\cos(\nu t)$, the period is $P = 2\pi/\nu$.

Example 3.5 GENERAL COSINE FUNCTION.

$$\begin{aligned} f(t) &= C\ \cos(\nu t+\theta) = C[\cos(\nu t)\cos(\theta)-\sin(\nu t)\sin(\theta)] \\ &= A\ \cos(\nu t)+B\ \sin(\nu t) \end{aligned} \tag{3.20}$$

where $C \geq 0$, $A = C\ \cos(\theta)$ and $B = -C\ \sin\ \theta$. Further,

$$C = \sqrt{A^2+B^2}, \quad \tan(\theta) = -B/A \text{ (if } C \neq 0). \tag{3.21}$$

In the above function, the different parameters have the following names:

$$\begin{aligned} C &= \text{amplitude}\,; \\ \nu &= \text{angular frequency (radians/time unit)}\,; \\ P &= 2\pi/\nu = \text{period}\,; \\ \bar{\nu} &= \frac{1}{P} = \frac{\nu}{2\pi} = \text{ frequency (number of cycles per time unit)}\,; \\ \theta &= \text{phase angle (usually } 0 \leq \theta < 2\pi \text{ or } -\pi/2 < \theta \leq \pi/2)\,. \end{aligned}$$

Example 3.6

$$\begin{aligned} f(t) &= C\ \sin(\nu t+\theta) = C\ \cos(\nu t+\theta-\pi/2) \\ &= C\left[\sin(\nu t)\cos(\theta)+\cos(\nu t)\sin(\theta)\right] \\ &= A\ \cos(\nu t)+B\ \sin(\nu t) \end{aligned} \tag{3.22}$$

where

$$0 \leq \nu < 2\pi\,, \tag{3.23}$$

$$A = C\ \sin(\theta) = C\ \cos\left(\theta-\frac{\pi}{2}\right)\,, \tag{3.24}$$

$$B = C\ \cos(\theta) = -C\ \sin\left(\theta-\frac{\pi}{2}\right)\,. \tag{3.25}$$

Consider the model

$$\begin{aligned} X_t &= C\ \cos(\nu t + \theta) \\ &= A\ \cos(\nu t) + B\ \sin(\nu t), t \in \mathbb{Z}. \end{aligned} \tag{3.26}$$

If A and B are constants,

$$\mathbb{E}(X_t) = A\ \cos(\nu t) + B\ \sin(\nu t)\,,\ t \in \mathbb{Z}, \tag{3.27}$$

so the process X_t is non-stationary (since the mean is not constant). Suppose now that A and B are random variables such that

$$\mathbb{E}(A) = \mathbb{E}(B) = 0, \quad \mathbb{E}(A^2) = \mathbb{E}(B^2) = \sigma^2, \quad \mathbb{E}(AB) = 0\,. \tag{3.28}$$

A and B do not depend on t but are fixed for each realization of the process $[A = A(\omega), B = B(\omega)]$. In this case,

$$\mathbb{E}(X_t) = 0\,, \tag{3.29}$$

$$\begin{aligned} \mathbb{E}(X_s X_t) &= \mathbb{E}(A^2)\cos(\nu s)\cos(\nu t) + \mathbb{E}(B^2)\sin(\nu s)\sin(\nu t) \\ &= \sigma^2[\cos(\nu s)\cos(\nu t) + \sin(\nu s)\sin(\nu t)] \\ &= \sigma^2\cos[\nu(t-s)]\,. \end{aligned} \tag{3.30}$$

The process X_t is stationary of order 2 with the following autocovariance and autocorrelation functions:

$$\gamma_X(k) = \sigma^2\cos(\nu k)\,, \tag{3.31}$$

$$\rho_X(k) = \cos(\nu k)\,. \tag{3.32}$$

If we add m cyclic processes of the form (3.26), we obtain a *harmonic process* of order m.

Definition 3.6 HARMONIC PROCESS OF ORDER m. *We say the process $\{X_t : t \in \mathbb{T}\}$ is a harmonic process of order m if it can written in the form*

$$X_t = \sum_{j=1}^{m} [A_j \cos(\nu_j t) + B_j \sin(\nu_j t)], \ \forall t \in \mathbb{T}, \tag{3.33}$$

where $\nu_1, \ldots, \nu_m$ are distinct constants in the interval $[0, 2\pi)$.

If A_j, B_j , $j = 1, \ldots, m$, are random variables in L_2 such that

$$\mathbb{E}(A_j) = \mathbb{E}(B_j) = 0, \ j = 1, \ldots, m \ , \tag{3.34}$$

$$\mathbb{E}(A_j^2) = \mathbb{E}(B_j^2) = \sigma_j^2, \ j = 1, \ldots, m \ , \tag{3.35}$$

$$\mathbb{E}(A_j A_k) = \mathbb{E}(B_j B_k) = 0, \text{ for } j \neq k, \tag{3.36}$$

$$\mathbb{E}(A_j B_k) = 0, \forall j, k \ , \tag{3.37}$$

the harmonic process X_t is second-order stationary, with:

$$\mathbb{E}(X_t) = 0 \ , \tag{3.38}$$

$$\mathbb{E}(X_s X_t) = \sum_{j=1}^{m} \sigma_j^2 \cos[\nu_j (t-s)] \ , \tag{3.39}$$

hence

$$\gamma_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) \ , \tag{3.40}$$

$$\rho_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) / \sum_{j=1}^{m} \sigma_j^2 \ . \tag{3.41}$$

If we add a white noise u_t to X_t in (3.33), we obtain again a second-order stationary process:

$$X_t = \sum_{j=1}^{m} [A_j \cos(\nu_j t) + B_j \sin(\nu_j t)] + u_t, \, t \in \mathbb{T} \ , \tag{3.42}$$

where the process $\{u_t : t \in \mathbb{T}\} \sim \mathrm{WN}(0, \sigma^2)$ is uncorrelated with A_j, B_j , $j = 1, \ldots, m$. In this case, $\mathbb{E}(X_t) = 0$ and

$$\gamma_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) + \sigma^2 \delta(k) \tag{3.43}$$

where

$$\begin{aligned} \delta(k) &= 1 \ \text{ if } k = 0 \\ &= 0 \ \text{ otherwise.} \end{aligned} \tag{3.44}$$

If a series can be described by an equation of the form (3.42), we can view it as a realization of a second-order stationary process.

3.3. Linear processes

Many stochastic processes with dependence are obtained through transformations of noise processes.

Definition 3.7 AUTOREGRESSIVE PROCESS. *The process $\{X_t : t \in \mathbb{T}\}$ is an autoregressive process of order p if it satisfies an equation of the form*

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t\,,\ \forall t \in \mathbb{T}\,, \tag{3.45}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\, \sigma^2)$. In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{AR}(p)\,. \tag{3.46}$$

Usually, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}_+$ (positive integers). If $\sum_{j=1}^{p} \varphi_j \neq 1$, we can define

$$\mu := \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j) \tag{3.47}$$

and write

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t\,,\ \forall t \in \mathbb{T}\,, \tag{3.48}$$

where $\tilde{X}_t := X_t - \mu$.

Definition 3.8 MOVING AVERAGE PROCESS. *The process $\{X_t : t \in \mathbb{T}\}$ is a moving average process of order q if it can written in the form*

$$X_t = \bar{\mu} + \sum_{j=0}^{q} \psi_j u_{t-j}\,, \forall t \in \mathbb{T}, \tag{3.49}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\, \sigma^2)$. In this case, we denote

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{MA}(q). \tag{3.50}$$

Without loss of generality, we can set $\psi_0 = 1$ and $\psi_j = -\theta_j$, $j = 1, \ldots, q$:

$$X_t = \bar{\mu} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}\ , t \in \mathbb{T} \tag{3.51}$$

or, equivalently,

$$\tilde{X}_t = u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{3.52}$$

where $\tilde{X}_t := X_t - \bar{\mu}$.

Definition 3.9 AUTOREGRESSIVE-MOVING-AVERAGE PROCESS. *The process* $\{X_t : t \in \mathbb{T}\}$ *is an autoregressive-moving-average (ARMA) process of order* (p, q) *if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}, \ \forall t \in \mathbb{T}, \tag{3.53}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$. *In this case, we denote*

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{ARMA}(p, q). \tag{3.54}$$

If $\sum_{j=1}^{p} \varphi_j \neq 1$, we can also write

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{3.55}$$

where $\tilde{X}_t = X_t - \mu$ and $\mu = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j)$.

Definition 3.10 MOVING AVERAGE PROCESS OF INFINITE ORDER. *The process* $\{X_t : t \in \mathbb{T}\}$ *is a moving-average process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j}\,,\ \forall t \in \mathbb{Z}, \tag{3.56}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\ \sigma^2)$. *We also say that* X_t *is a* weakly linear process. *In this case, we denote*

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{MA}(\infty). \tag{3.57}$$

In particular, if $\psi_j = 0$ *for* $j < 0$, i.e.

$$X_t = \bar{\mu} + \sum_{j=0}^{\infty} \psi_j u_{t-j}\,,\ \forall t \in \mathbb{Z}, \tag{3.58}$$

we say that X_t *is a causal function of* u_t *(causal linear process).*

Definition 3.11 AUTOREGRESSIVE PROCESS OF INFINITE ORDER. *The process* $\{X_t : t \in \mathbb{T}\}$ *is an autoregressive process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=1}^{\infty} \varphi_j X_{t-j} + u_t\,,\ t \in \mathbb{T}, \tag{3.59}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\ \sigma^2)$. *In this case, we denote*

$$\{X_t : t \in \mathbb{T}\} \sim \mathrm{AR}(\infty). \tag{3.60}$$

Definition 3.12 Remark 3.1 We can generalize the notions defined above by assuming that $\{u_t : t \in \mathbb{Z}\}$ is a noise. Unless stated otherwise, we will suppose $\{u_t\}$ is a white noise.

QUESTIONS :

(1) Under which conditions are the processes defined above stationary (strictly or in L_r)?

(2) Under which conditions are the processes $\mathrm{MA}(\infty)$ or $\mathrm{AR}(\infty)$ well defined (convergent series)?

(3) What are the links between the different classes of processes defined above?

(4) When a process is stationary, what are its autocovariance and autocorrelation functions?

Definition 3.13 RANDOM WALK. *The process $\{X_t : t \in \mathbb{T}\}$ is a random walk if it satisfies an equation of the form*

$$X_t - X_{t-1} = v_t,\ \forall t \in \mathbb{T}, \tag{3.61}$$

where $\{v_t : t \in \mathbb{T}\} \sim$ IID. To ensure that this process is well defined, we suppose that $n_0 \neq -\infty$. If $n_0 = -1$, we can write

$$X_t = X_0 + \sum_{j=1}^{t} v_j \tag{3.62}$$

hence the name "integrated process". If $\mathbb{E}(v_t) = \bar{\mu}$ or $\mathrm{Med}(v_t) = \bar{\mu}$, one often writes

$$X_t - X_{t-1} = \bar{\mu} + u_t \tag{3.63}$$

where $u_t := v_t - \bar{\mu} \sim$ IID and $\mathbb{E}(u_t) = 0$ or $\mathrm{Med}(u_t) = 0$ (depending on whether $\mathbb{E}(u_t) = 0$ or $\mathrm{Med}(u_t) = 0$). If $\bar{\mu} \neq 0$, we say the the random walk has a drift.

Definition 3.14 WEAK RANDOM WALK. *The process* $\{X_t : t \in \mathbb{T}\}$ *is a* weak random walk *if* X_t *satisfies an equation of the form*

$$X_t - X_{t-1} = \bar{\mu} + u_t \tag{3.64}$$

where $\{u_t : t \in \mathbb{T}\} \sim \mathrm{WN}(0,\, \sigma^2)$, $\{u_t : t \in \mathbb{T}\} \sim \mathrm{WN}(0,\, \sigma_t^2)$, *or* $\{u_t : t \in \mathbb{T}\} \sim IND(0)]$.

Definition 3.15 INTEGRATED PROCESS. *The process* $\{X_t : t \in \mathbb{T}\}$ *is integrated of order* d *if it can be written in the form*

$$(1-B)^d X_t = Z_t \ , \forall t \in \mathbb{T}, \tag{3.65}$$

where $\{Z_t : t \in \mathbb{T}\}$ *is a stationary process (usually stationary of order 2) and* d *is a non-negative integer* $(d = 0, 1, 2, ...)$. *In particular, if* $\{Z_t : t \in \mathbb{T}\}$ *is an* $\mathrm{AR}MA(p,q)$ *stationary process,* $\{X_t : t \in \mathbb{T}\}$ *is an* $\mathrm{ARIMA}(p,d,q)$ *process*: $\{X_t : t \in \mathbb{T}\} \sim \mathrm{ARIMA}(p,d,q)$. *We note*

$$\begin{aligned}
B\, X_t &= X_{t-1} \ , && (3.66)\\
(1-B)X_t &= X_t - X_{t-1} \ , && (3.67)\\
(1-B)^2 X_t &= (1-B)(1-B)X_t = (1-B)(X_t - X_{t-1}) && (3.68)\\
&= X_t - 2X_{t-1} + X_{t-2}, && (3.69)\\
(1-B)^d X_t &= (1-B)(1-B)^{d-1} X_t, d = 1, 2, ... && (3.70)
\end{aligned}$$

where $(1-B)^0 = 1$.

3.5. Deterministic trends

Definition 3.16 DETERMINISTIC TREND. *The process $\{X_t : t \in \mathbb{T}\}$ follows a deterministic trend if it can be written in the form*

$$X_t = f(t) + Z_t \ , \forall t \in \mathbb{T}, \tag{3.71}$$

where $f(t)$ is a deterministic function of time and $\{Z_t : t \in \mathbb{T}\}$ is a noise or a stationary process.

Example 3.7 Important cases of deterministic trend:

$$X_t = \beta_0 + \beta_1 t + u_t, \tag{3.72}$$

$$X_t = \sum_{j=0}^{k} \beta_j t^j + u_t, \tag{3.73}$$

where $\{u_t : t \in \mathbb{T}\} \sim \mathrm{WN}(0, \sigma^2)$.

4. Transformations of stationary processes

Theorem 4.1 ABSOLUTE MOMENT SUMMABILITY CRITERION FOR CONVERGENCE OF A LINEAR TRANSFORMATION OF A STOCHASTIC PROCESS. *Let $\{X_t : t \in \mathbb{Z}\}$ be a stochastic process on the integers, $r \geq 1$ and $\{a_j : j \in \mathbb{Z}\}$ a sequence of real numbers. If*

$$\sum_{j=-\infty}^{\infty} |a_j| \, \mathbb{E}(|X_{t-j}|^r)^{1/r} < \infty \tag{4.1}$$

then, for any t, the random series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely a.s. and in mean of order r to a random variable Y_t such that $\mathbb{E}(|Y_t|^r) < \infty$.

PROOF. See Dufour (2016). □

Theorem 4.2 ABSOLUTE SUMMABILITY CRITERION FOR CONVERGENCE OF A LINEAR TRANSFORMATION OF A WEAKLY STATIONARY PROCESS. *Let $\{X_t : t \in \mathbb{Z}\}$ be a second-order stationary process and $\{a_j : j \in \mathbb{Z}\}$ an sequence of real numbers absolutely convergent sequence of real numbers,* i.e.

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty . \tag{4.2}$$

Then the random series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely a.s. and in mean of order 2 to a random variable $Y_t \in L_2$, $\forall t$, and the process $\{Y_t : t \in \mathbb{Z}\}$ is second-order stationary with autocovariance function

$$\gamma_Y(k) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \gamma_X(k-i+j) . \tag{4.3}$$

PROOF. See Gouriéroux and Monfort (1997, Property 5.6). □

Theorem 4.3 NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE OF LINEAR FILTERS OF ARBITRARY WEAKLY STATIONARY PROCESSES. *The series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely a.s. for any second-order stationary process $\{X_t : t \in \mathbb{Z}\}$ iff*

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty . \tag{4.4}$$

5. Infinite order moving averages

We study here random series of the form

$$\sum_{j=o}^{\infty} \psi_j u_{t-j},\, t \in \mathbb{Z} \tag{5.1}$$

and

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j},\, t \in \mathbb{Z} \tag{5.2}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$.

5.1. Convergence conditions

Theorem 5.1 MEAN SQUARE CONVERGENCE OF AN INFINITE MOVING AVERAGE. *Let $\{\psi_j : j \in \mathbb{Z}\}$ be a sequence of fixed real constants and $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$.*

(1) *If $\sum_{j=1}^{\infty} \psi_j^2 < \infty$, $\sum_{j=1}^{\infty} \psi_j u_{t-j}$ converges in q.m. to a random variable X_{Ut} in L_2.*

(2) *If $\sum_{j=-\infty}^{0} \psi_j^2 < \infty$, $\sum_{j=-\infty}^{0} \psi_j u_{t-j}$ converges in q.m. to a random variable X_{Lt} in L_2.*

(3) *If $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in q.m. to a random variable X_t in L_2, and $\sum_{j=-n}^{n} \psi_j u_{t-j} \xrightarrow[n\to\infty]{2} X_t$.*

PROOF. Suppose $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. We can write

$$\sum_{j=1}^{\infty} \psi_j u_{t-j} = \sum_{j=1}^{\infty} Y_j(t)\,, \qquad \sum_{j=-\infty}^{0} \psi_j u_{t-j} = \sum_{j=-\infty}^{0} Y_j(t) \tag{5.3}$$

where $Y_j(t) := \psi_j u_{t-j}$,

$$\mathbb{E}\big[Y_j(t)^2\big] = \psi_j^2\, \mathbb{E}(u_{t-j}^2) = \psi_j^2\, \sigma^2 < \infty, \text{ for } t \in \mathbb{Z},$$

and the variables $Y_j(t)$, $t \in \mathbb{Z}$, are orthogonal. If $\sum_{j=1}^{\infty} \psi_j^2 < \infty$, the series $\sum_{j=1}^{\infty} Y_j(t)$ converges in q.m. to a random variable X_{Ut} such that $\mathbb{E}\big[X_{Ut}^2\big] < \infty$, i.e.

$$\sum_{j=1}^{n} Y_j(t) \underset{n\to\infty}{\overset{2}{\to}} X_{Ut} := \sum_{j=1}^{\infty} \psi_j u_{t-j}\,; \tag{5.4}$$

see Dufour (2016, Section on "Series of orthogonal variables"). By a similar argument, if $\sum_{j=-\infty}^{0} \psi_j^2 < \infty$, the series $\sum_{j=-\infty}^{0} Y_j(t)$ converges in q.m. to a random variable X_{Ut} such that $\mathbb{E}\big[X_{Lt}^2\big] < \infty$, i.e.

$$\sum_{j=-m}^{0} Y_j(t) \underset{m\to\infty}{\overset{2}{\to}} X_{Lt} := \sum_{j=-\infty}^{0} \psi_j u_{t-j}\,. \tag{5.5}$$

Finally, if $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, we must have $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ and $\sum_{j=-\infty}^{0} \psi_j^2 < \infty$, hence

$$\sum_{j=-m}^{n} Y_j(t) = \sum_{j=-m}^{0} Y_j(t) + \sum_{j=1}^{n} Y_j(t) \underset{\substack{m\to\infty \\ n\to\infty}}{\overset{2}{\to}} X_{Lt} + X_{Ut} := X_t := \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \tag{5.6}$$

where, by the c_r-inequality [see Dufour (2022)],

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[(X_{Lt} + X_{Ut})^2\right] \leq 2\{\mathbb{E}\left[X_{Lt}^2\right] + \mathbb{E}\left[X_{Ut}^2\right]\} < \infty\,. \tag{5.7}$$

The random variable X_t is denoted:

$$X_t := \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}\,. \tag{5.8}$$

The last statement on the convergence of $\sum_{j=-n}^{n} \psi_j u_{t-j}$ follows from the definition of mean-square convergence of $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$. □

Corollary 5.2 ALMOST SURE CONVERGENCE OF AN INFINITE MOVING AVERAGE. *Let* $\{\psi_j : j \in \mathbb{Z}\}$ *be a sequence of fixed real constants, and* $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$.

(1) *If* $\sum_{j=1}^{\infty} |\psi_j| < \infty$, $\sum_{j=1}^{\infty} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_{Ut} *in* L_2.

(2) *If* $\sum_{j=-\infty}^{0} |\psi_j| < \infty$, $\sum_{j=-\infty}^{0} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_{Lt} *in* L_2.

(3) *If* $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_t *in* L_2, $\sum_{j=-n}^{n} \psi_j u_{t-j} \xrightarrow[n\to\infty]{a.s.} X_t$ *and* $\sum_{j=-n}^{n} \psi_j u_{t-j} \xrightarrow[n\to\infty]{2} X_t$.

PROOF. This result from Theorem 5.1 and the observation that

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty. \tag{5.9}$$

□

Theorem 5.3 ALMOST SURE CONVERGENCE OF AN INFINITE MOVING AVERAGE OF INDEPENDENT VARIABLES. *Let* $\{\psi_j : j \in \mathbb{Z}\}$ *be a sequence of fixed real constants, and* $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{IID}(0, \sigma^2)$.

(1) *If* $\sum_{j=1}^{\infty} \psi_j^2 < \infty$, $\sum_{j=1}^{\infty} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_{Ut} *in* L_2.

(2) *If* $\sum_{j=-\infty}^{0} \psi_j^2 < \infty$, $\sum_{j=-\infty}^{0} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_{Lt} *in* L_2.

(3) *If* $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ *converges a.s. and in q.m. to a random variable* X_t *in* L_2, $\sum_{j=-n}^{n} \psi_j u_{t-j} \xrightarrow[n\to\infty]{a.s.} X_t$ *and* $\sum_{j=-n}^{n} \psi_j u_{t-j} \xrightarrow[n\to\infty]{2} X_t$.

PROOF. This result from Theorem 5.1 and by applying results on the convergence of series of independent variables [Dufour (2016, Section on "Series of independent variables")]. □

5.2. Mean, variance and covariances

Let

$$S_n(t) = \sum_{j=-n}^{n} \psi_j u_{t-j}\,. \tag{5.10}$$

By Theorem 5.1, we have:

$$S_n(t) \xrightarrow[n\to\infty]{2} X_t \tag{5.11}$$

where $X_t \in L_2$, hence [using Dufour (2016, Section on "Convergence of functions of random variables")]

$$\mathbb{E}(X_t) = \lim_{n\to\infty} \mathbb{E}[S_n(t)] = 0\,, \tag{5.12}$$

$$\mathrm{V}(X_t) = \mathbb{E}(X_t^2) = \lim_{n\to\infty} \mathbb{E}[S_n(t)^2] = \lim_{n\to\infty} \sum_{j=-n}^{n} \psi_j^2 \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j^2\,, \tag{5.13}$$

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+k}) &= \mathbb{E}(X_t X_{t+k}) = \lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=-n}^{n} \psi_i u_{t-i}\right)\left(\sum_{j=-n}^{n} \psi_j u_{t+k-j}\right)\right] \\
&= \lim_{n\to\infty} \sum_{i=-n}^{n} \sum_{j=-n}^{n} \psi_i \psi_j \mathbb{E}(u_{t-i} u_{t+k-j}) \\
&= \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{i=-n}^{n-k} \psi_i \psi_{i+k} \sigma^2 = \sigma^2 \sum\limits_{i=-\infty}^{\infty} \psi_i \psi_{i+k}, & \text{if } k \geq 1, \\ \lim\limits_{n\to\infty} \sum\limits_{j=-n}^{n} \psi_j \psi_{j+|k|} \sigma^2 = \sigma^2 \sum\limits_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}\,, & \text{if } k \leq -1, \end{cases}
\end{aligned} \tag{5.14}$$

since $t-i=t+k-j \Rightarrow j=i+k$ and $i=j-k$. For any $k \in \mathbb{Z}$, we can write

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t+k}) &= \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}\,, \qquad (5.15) \\ \mathrm{Corr}(X_t, X_{t+k}) &= \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|} \Big/ \sum_{j=-\infty}^{\infty} \psi_j^2\,. \qquad (5.16) \end{aligned}$$

The series $\sum\limits_{j=-\infty}^{\infty} \psi_j \psi_{j+k}$ converges absolutely, for

$$\left| \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \right| \leq \sum_{j=-\infty}^{\infty} \left| \psi_j \psi_{j+k} \right| \leq \left[\sum_{j=-\infty}^{\infty} \psi_j^2 \right]^{\frac{1}{2}} \left[\sum_{j=-\infty}^{\infty} \psi_{j+k}^2 \right]^{\frac{1}{2}} < \infty\,. \qquad (5.17)$$

If $X_t = \mu + X_t = \mu + \sum\limits_{j=-\infty}^{+\infty} \psi_j u_{t-j}$, then

$$\mathbb{E}(X_t) = \mu\,,\ \mathrm{Cov}(X_t, X_{t+k}) = \mathrm{Cov}(X_t, X_{t+k})\,. \qquad (5.18)$$

In the case of a causal MA(∞) process causal, we have

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \qquad (5.19)$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$,

$$\mathrm{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} \ , \tag{5.20}$$

$$\mathrm{Corr}(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} \Big/ \sum_{j=0}^{\infty} \psi_j^2 \ . \tag{5.21}$$

5.3. Stationarity

The process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \ , t \in \mathbb{Z}, \tag{5.22}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, is second-order stationary, for $\mathbb{E}(X_t)$ and $\mathrm{Cov}(X_t, X_{t+k})$ do not depend on t. If we suppose that $\{u_t : t \in \mathbb{Z}\} \sim$IID, with $\mathbb{E}|u_t| < \infty$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, the process is strictly stationary.

5.4. Operational notation

We can denote the process MA(∞)

$$X_t = \mu + \psi(B)u_t = \mu + \left(\sum_{j=-\infty}^{\infty} \psi_j B^j \right) u_t \tag{5.23}$$

where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$ and $B^j u_t = u_{t-j}$.

6. Finite order moving averages

The MA(q) process can be written

$$X_t = \mu + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{6.1}$$

where $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. This process is a special case of the MA(∞) process with

$$\begin{aligned} \psi_0 &= 1\,,\ \psi_j = -\theta_j\,,\ \text{for } 1 \le j \le q\,, \\ \psi_j &= 0\,,\ \text{for } j < 0 \text{ or } j > q. \end{aligned} \tag{6.2}$$

This process is clearly second-order stationary, with

$$\mathbb{E}(X_t) = \mu\,, \tag{6.3}$$

$$\mathrm{V}(X_t) = \sigma^2 \left(1 + \sum_{j=1}^{q} \theta_j^2 \right)\,, \tag{6.4}$$

$$\gamma(k) \ :\ = \mathrm{Cov}(X_t, X_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}\,. \tag{6.5}$$

On defining $\theta_0 := -1$, we then see that

$$\gamma(k) = \sigma^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k}$$

$$
\begin{aligned}
&= \sigma^2 \left[-\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right] && (6.6)\\
&= \sigma^2 [-\theta_k + \theta_1 \theta_{k+1} + \ldots + \theta_{q-k} \theta_q] \,, \text{ for } 1 \le k \le q, && (6.7)
\end{aligned}
$$

$$
\begin{aligned}
\gamma(k) &= 0 \,, \text{ for } k \ge q+1, \\
\gamma(-k) &= \gamma(k) \,, \text{ for } k < 0 \,. && (6.8)
\end{aligned}
$$

The autocorrelation function of X_t is thus

$$
\begin{aligned}
\rho(k) &= \left(-\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k} \right) / \left(1 + \sum_{j=1}^{q} \theta_j^2 \right) \,, && \text{for } 1 \le k \le q \\
&= 0 \,, && \text{for } k \ge q+1
\end{aligned} \tag{6.9}
$$

The autocorrelations are zero for $k \ge q+1$.

For $q = 1$,

$$\begin{aligned} \rho(k) &= -\theta_1/(1+\theta_1^2), \quad \text{if } k = 1\ , \\ &= 0\,, \qquad\qquad\qquad\quad \text{if } k \geq 2, \end{aligned} \tag{6.10}$$

hence $|\rho(1)| \leq 0.5$. For $q = 2$,

$$\begin{aligned} \rho(k) &= (-\theta_1 + \theta_1\theta_2)/(1+\theta_1^2+\theta_2^2)\,, \quad \text{if } k = 1\,, \\ &= -\theta_2/(1+\theta_1^2+\theta_2^2)\,, \qquad\qquad \text{if } k = 2\ , \\ &= 0\,, \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } k \geq 3\,, \end{aligned} \tag{6.11}$$

hence $|\rho(2)| \leq 0.5$.

For any MA(q) process,

$$\rho(q) = -\theta_q/(1+\theta_1^2+\ldots+\theta_q^2)\,, \tag{6.12}$$

hence $|\rho(q)| \leq 0.5$.

There are general constraints on the autocorrelations of an MA(q) process:

$$|\rho(k)| \leq \cos(\pi/\{[q/k]+2\}) \tag{6.13}$$

where $[x]$ is the largest integer less than or equal to x. From the latter formula, we see:

$$\begin{aligned}
\text{for } q = 1\,, \quad & |\rho(1)| \leq \cos(\pi/3) = 0.5, \\
\text{for } q = 2\,, \quad & |\rho(1)| \leq \cos(\pi/4) = 0.7071, \\
& |\rho(2)| \leq \cos(\pi/3) = 0.5, \\
\text{for } q = 3\,, \quad & |\rho(1)| \leq \cos(\pi/5) = 0.809, \\
& |\rho(2)| \leq \cos(\pi/3) = 0.5, \\
& |\rho(3)| \leq \cos(\pi/3) = 0.5.
\end{aligned} \tag{6.14}$$

See Chanda (1962), and Kendall, Stuart, and Ord (1983, p. 519).

7. Autoregressive processes

Consider a process $\{X_t : t \in \mathbb{Z}\}$ which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t, \ \forall t \in \mathbb{Z}, \tag{7.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$. In symbolic notation,

$$\varphi(B) X_t = \bar{\mu} + u_t, \ t \in \mathbb{Z}, \tag{7.2}$$

where $\varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$.

7.1. Stationarity

Consider the process AR(1)

$$X_t = \varphi_1 X_{t-1} + u_t,\ \varphi_1 \neq 0. \tag{7.3}$$

If X_t is S2 ,

$$\mathbb{E}(X_t) = \varphi_1 \mathbb{E}(X_{t-1}) = \varphi_1 \mathbb{E}(X_t) \tag{7.4}$$

hence $\mathbb{E}(X_t) = 0$. By successive substitutions,

$$\begin{aligned} X_t &= \varphi_1[\varphi_1 X_{t-2} + u_{t-1}] + u_t \\ &= u_t + \varphi_1 u_{t-1} + \varphi_1^2 X_{t-2} \\ &= \sum_{j=0}^{N-1} \varphi_1^j u_{t-j} + \varphi_1^N X_{t-N}\ . \end{aligned} \tag{7.5}$$

If we suppose that X_t is S2 with $\mathbb{E}(X_t^2) \neq 0$, we see that

$$\mathbb{E}\left[\left(X_t - \sum_{j=0}^{N-1} \varphi_1^j u_{t-j}\right)^2\right] = \varphi_1^{2N}\mathbb{E}(X_{t-N}^2) = \varphi_1^{2N}\mathbb{E}(X_t^2) \underset{N\to\infty}{\longrightarrow} 0 \Leftrightarrow |\varphi_1| < 1. \tag{7.6}$$

The series $\sum\limits_{j=0}^{\infty} \varphi_1^j u_{t-j}$ converges in *q.m.* to

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} := (1-\varphi_1 B)^{-1} u_t = \frac{1}{1-\varphi_1 B}\, u_t \tag{7.7}$$

where

$$(1-\varphi_1 B)^{-1} = \sum_{j=0}^{\infty} \varphi_1^j B^j \,. \tag{7.8}$$

Since

$$\sum_{j=0}^{\infty} \mathbb{E}|\varphi_1^j u_{t-j}| \leq \sigma \sum_{j=0}^{\infty} |\varphi_1|^j = \frac{\sigma}{1-|\varphi_1|} < \infty \tag{7.9}$$

when $|\varphi_1| < 1$, the convergence is also a.s. The process $X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j}$ is S2.

When $|\varphi_1| < 1$, the difference equation

$$(1-\varphi_1 B)X_t = u_t \tag{7.10}$$

has a unique stationary solution which can be written

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} = (1-\varphi_1 B)^{-1} u_t. \tag{7.11}$$

The latter is thus a causal MA(∞) process.

This condition is sufficient (but non necessary) for the existence of a unique stationary solution. The stationarity condition is often expressed by saying that the polynomial $\varphi(z) = 1 - \varphi_1 z$ has all its roots outside the unit circle $|z| = 1$:

$$1 - \varphi_1 z_* = 0 \Leftrightarrow z_* = \frac{1}{\varphi_1} \tag{7.12}$$

where $|z_*| = 1/|\varphi_1| > 1$. In this case, we also have $\mathbb{E}(X_{t-k}u_t) = 0$, $\forall k \geq 1$. The same conclusion holds if we consider the general process

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t \ . \tag{7.13}$$

For the AR(p) process,

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t \tag{7.14}$$

or

$$\varphi(B) X_t = \bar{\mu} + u_t, \tag{7.15}$$

the stationarity condition is the following:

$$\begin{gathered}\text{if the polynomial } \varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p \text{ has all its roots outside the unit circle,} \\ \text{the equation (7.14) has one and only one weakly stationary solution.}\end{gathered} \tag{7.16}$$

$\varphi(z)$ is a polynomial of order p with no root equal to zero. It can be written in the form

$$\varphi(z) = (1 - G_1 z)(1 - G_2 z) \ldots (1 - G_p z)\,, \tag{7.17}$$

so the roots of $\varphi(z)$ are

$$z_1^* = 1/G_1, \ldots, z_p^* = 1/G_p\,, \tag{7.18}$$

and the stationarity condition have the equivalent form:

$$|G_j| < 1,\ j = 1, \ldots, p. \tag{7.19}$$

The stationary solution can be written

$$X_t = \varphi(B)^{-1}\bar{\mu} + \varphi(B)^{-1}u_t = \mu + \varphi(B)^{-1}u_t \tag{7.20}$$

where

$$\mu = \bar{\mu} / \left(1 - \sum_{j=1}^{p} \varphi_j\right), \tag{7.21}$$

$$\begin{aligned} \varphi(B)^{-1} &= \prod_{j=1}^{p} (1 - G_j B)^{-1} = \prod_{j=1}^{p} \left(\sum_{k=0}^{\infty} G_j^k B^k\right) \\ &= \sum_{j=1}^{p} \frac{K_j}{1 - G_j B} \end{aligned} \tag{7.22}$$

and $K_1, \ldots, K_p$ are constants (expansion in partial fractions). Consequently,

$$\begin{aligned} X_t &= \mu + \sum_{j=1}^{p} \left(\frac{K_j}{1 - G_j B}\right) u_t \\ &= \mu + \sum_{k=0}^{\infty} \psi_k u_{t-k} = \mu + \psi(B) u_t \end{aligned} \tag{7.23}$$

where $\psi_k = \sum_{j=1}^{p} K_j G_j^k$. Thus

$$\mathbb{E}(X_{t-j} u_t) = 0, \forall j \geq 1. \tag{7.24}$$

For the process AR(1) and AR(2), the stationarity conditions can be written as follows.

(a) AR(1) – For $(1-\varphi_1 B)X_t = \bar{\mu} + u_t$,

$$|\varphi_1| < 1 \tag{7.25}$$

(b) AR(2) – For $(1-\varphi_1 B - \varphi_2 B^2)X_t = \bar{\mu} + u_t$,

$$\varphi_2 + \varphi_1 < 1 \tag{7.26}$$

$$\varphi_2 - \varphi_1 < 1 \tag{7.27}$$

$$-1 < \varphi_2 < 1 \tag{7.28}$$

7.2. Mean, variance and autocovariances

Suppose:

$$\begin{aligned} &\text{a) the autoregressive process } X_t \text{ is second-order stationary with } \sum_{j=1}^{p} \varphi_j \neq 1 \\ &\text{and} \\ &\text{b) } \mathbb{E}(X_{t-j} u_t) = 0 \text{ , } \forall j \geq 1 \text{ ,} \end{aligned} \tag{7.29}$$

i.e., we assume that X_t is a weakly stationary solution of the equation (7.14) such that $\mathbb{E}(X_{t-j} u_t) = 0, \forall j \geq 1$.

By the stationarity assumption, we have: $\mathbb{E}(X_t) = \mu, \forall t$, hence

$$\mu = \bar{\mu} + \sum_{j=1}^{p} \varphi_j \mu \tag{7.30}$$

and

$$\mathbb{E}(X_t) = \mu = \bar{\mu} / \left(1 - \sum_{j=1}^{p} \varphi_j \right) . \tag{7.31}$$

For stationarity to hold, it is necessary that $\sum_{j=1}^{p} \varphi_j \neq 1$. Let us rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t \tag{7.32}$$

where $\tilde{X}_t = X_t - \mu$, $\mathbb{E}(\tilde{X}_t) = 0$. Then, for $k \geq 0$,

$$\tilde{X}_{t+k} = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t+k-j} + u_{t+k}\,, \tag{7.33}$$

$$\mathbb{E}(\tilde{X}_{t+k}\,\tilde{X}_t) = \sum_{j=1}^{p} \varphi_j \mathbb{E}(\tilde{X}_{t+k-j}\tilde{X}_t) + \mathbb{E}(u_{t+k}\tilde{X}_t)\,, \tag{7.34}$$

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) + \mathbb{E}(u_{t+k}\,\tilde{X}_t)\,, \tag{7.35}$$

where

$$\begin{aligned} \mathbb{E}(u_{t+k}\,\tilde{X}_t) &= \sigma^2, \text{ if } k = 0, \\ &= 0\,, \text{ if } k \geq 1. \end{aligned} \tag{7.36}$$

Thus

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j), k \geq 1. \tag{7.37}$$

These formulae are called the "Yule-Walker equations". If we know $\rho(0), \ldots, \rho(p-1)$, we can easily compute $\rho(k)$ for $k \geq p+1$. We can also write the Yule-Walker equations in the form:

$$\varphi(B)\rho(k) = 0, \text{ for } k \geq 1, \tag{7.38}$$

where $B^j \rho(k) := \rho(k-j)$. To obtain $\rho(1), \ldots, \rho(p-1)$ for $p > 1$, it is sufficient to solve the linear equation

system:

$$\begin{aligned}
\rho(1) &= \varphi_1 + \varphi_2\rho(1) + \cdots + \varphi_p\rho(p-1) \\
\rho(2) &= \varphi_1\rho(1) + \varphi_2 + \cdots + \varphi_p\rho(p-2) \\
&\vdots \\
\rho(p-1) &= \varphi_1\rho(p-2) + \varphi_2\rho(p-3) + \cdots + \varphi_p\rho(1)
\end{aligned} \tag{7.39}$$

where we use the identity $\rho(-j) = \rho(j)$. The other autocorrelations may then be obtained by recurrence:

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j),\ k \geq p\,. \tag{7.40}$$

To compute $\gamma(0) = V(X_t)$, we solve the equation

$$\begin{aligned}
\gamma(0) &= \sum_{j=1}^{p} \varphi_j \gamma(-j) + \mathbb{E}(u_t\, \tilde{X}_t) \\
&= \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2
\end{aligned} \tag{7.41}$$

hence, using $\gamma(j) = \rho(j)\gamma(0)$,

$$\gamma(0)\left[1 - \sum_{j=1}^{p} \varphi_j \rho(j)\right] = \sigma^2 \tag{7.42}$$

and

$$\gamma(0) = \frac{\sigma^2}{1 - \sum_{j=1}^{p} \varphi_j \rho(j)}. \tag{7.43}$$

7.3. Special cases

1. AR(1) – If

$$\tilde{X}_t = \varphi_1 \tilde{X}_{t-1} + u_t \tag{7.44}$$

we have:

$$\begin{aligned}
\rho(1) &= \varphi_1 \,, && (7.45)\\
\rho(k) &= \varphi_1 \rho(k-1)\,, \text{ for } k \geq 1\,, && (7.46)\\
\rho(2) &= \varphi_1 \rho(1) = \varphi_1^2\,, && (7.47)\\
\rho(k) &= \varphi_1^k, k \geq 1\,, && (7.48)\\
\gamma(0) &= \mathrm{V}(X_t) = \frac{\sigma^2}{1-\varphi_1^2}\,. && (7.49)
\end{aligned}$$

These is no constraint on $\rho(1)$, but there are constraints on $\rho(k)$ for $k \geq 2$.

2. AR(2) – If

$$X_t = \varphi_1 \tilde{X}_{t-1} + \varphi_2 \tilde{X}_{t-2} + u_t\,, \tag{7.50}$$

we have:

$$\begin{aligned}
\rho(1) &= \varphi_1 + \varphi_2 \rho(1)\,, && (7.51)\\
\rho(1) &= \frac{\varphi_1}{1-\varphi_2}\,, && (7.52)
\end{aligned}$$

$$\rho(2) = \frac{\varphi_1^2}{1-\varphi_2} + \varphi_2 = \frac{\varphi_1^2 + \varphi_2(1-\varphi_2)}{1-\varphi_2}, \tag{7.53}$$

$$\rho(k) = \varphi_1 \rho(k-1) + \varphi_2 \rho(k-2), \text{ for } k \geq 2. \tag{7.54}$$

Constraints on $\rho(1)$ and $\rho(2)$ are entailed by the stationarity of the AR(2) model:

$$|\rho(1)| < 1, |\rho(2)| < 1, \tag{7.55}$$

$$\rho(1)^2 < \frac{1}{2}[1+\rho(2)]; \tag{7.56}$$

see Box and Jenkins (1976, p. 61).

7.4. Explicit form for the autocorrelations

The autocorrelations of an AR(p) process satisfy the equation

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j),\, k \geq 1, \tag{7.57}$$

where $\rho(0) = 1$ and $\rho(-k) = \rho(k)$, or equivalently

$$\varphi(B)\rho(k) = 0\,,\ k \geq 1. \tag{7.58}$$

The autocorrelations can be obtained by solving the homogeneous difference equation (7.57).

The polynomial $\varphi(z)$ has m distinct non-zero roots $z_1^*, \ldots, z_m^*$ (where $1 \leq m \leq p$) with multiplicities $p_1, \ldots, p_m$ (where $\sum_{j=1}^{m} p_j = p$), so that $\varphi(z)$ can be written

$$\varphi(z) = (1 - G_1 z)^{p_1}(1 - G_2 z)^{p_2} \cdots (1 - G_m z)^{p_m} \tag{7.59}$$

where $G_j = 1/z_j^*$, $j = 1, \ldots, m$. The roots are real or complex numbers. If z_j^* is a complex (non real) root, its conjugate $\bar{z}_j^*$ is also a root. Consequently, the solutions of equation (7.57) have the general form

$$\rho(k) = \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j - 1} A_{j\ell} k^{\ell} \right) G_j^k,\, k \geq 1, \tag{7.60}$$

where the $A_{j\ell}$ are (possibly complex) constants which can be determined from the values p autocorrelations. We can easily find $\rho(1), \ldots, \rho(p)$ from the Yule-Walker equations.

If we write $G_j = r_j e^{i\theta_j}$, where $i = \sqrt{-1}$ while r_j and θ_j are real numbers $(r_j > 0)$, we see that

$$\begin{aligned}
\rho(k) &= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j-1} A_{j\ell} k^{\ell} \right) r_j^k e^{i\theta_j k} \\
&= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j-1} A_{j\ell} k^{\ell} \right) r_j^k [\cos(\theta_j k) + i \ \sin(\theta_j k)] \\
&= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j-1} A_{j\ell} k^{\ell} \right) r_j^k \cos(\theta_j k). \qquad (7.61)
\end{aligned}$$

By stationarity, $0 < |G_j| = r_j < 1$ so that $\rho(k) \to 0$ when $k \to \infty$. The autocorrelations decrease at an exponential rate with oscillations.

7.5. MA(∞) representation of an AR(p) process

We have seen that a weakly stationary process which satisfies the equation

$$\varphi(B)\tilde{X}_t = u_t \tag{7.62}$$

where $\varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$, can be written as

$$\tilde{X}_t = \psi(B)u_t \tag{7.63}$$

with

$$\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j \tag{7.64}$$

To compute the coefficients ψ_j, it is sufficient to note that

$$\varphi(B)\psi(B) = 1. \tag{7.65}$$

Setting $\psi_j = 0$ for $j < 0$, we see that

$$\begin{aligned}\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right)\left(\sum_{j=-\infty}^{\infty} \psi_j B^j\right) &= \sum_{j=-\infty}^{\infty} \psi_j \left(B^j - \sum_{k=1}^{p} \varphi_k B^{j+k}\right) \\ &= \sum_{j=-\infty}^{\infty} \left(\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k}\right) B^j = \sum_{j=-\infty}^{\infty} \tilde{\psi}_j B^j = 1. \end{aligned} \tag{7.66}$$

Thus $\tilde{\psi}_j = 1$, if $j = 0$, and $\tilde{\psi}_j = 0$, if $j \neq 0$. Consequently,

$$\begin{aligned} \varphi(B)\psi_j = \psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} &= 1\,, \text{ if } j = 0 \\ &= 0\,, \text{ if } j \neq 0, \end{aligned} \tag{7.67}$$

where $B^k \psi_j := \psi_{j-k}$. Since $\psi_j = 0$ for $j < 0$, we see that:

$$\begin{aligned} \psi_0 &= 1\,, \\ \psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}\,, \text{ for } j \geq 1. \end{aligned} \tag{7.68}$$

More explicitly,

$$\begin{aligned} \psi_0 &= 1\,, \\ \psi_1 &= \varphi_1 \psi_0 = \varphi_1\,, \\ \psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 = \varphi_1^2 + \varphi_2\,, \\ \psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3 = \varphi_1^3 + 2\,\varphi_2 \varphi_1 + \varphi_3\,, \\ &\vdots \\ \psi_p &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}\,, \\ \psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, \; j \geq p+1\,. \end{aligned} \tag{7.69}$$

Under the stationarity condition *i.e.*, the roots of $\varphi(z) = 0$ are outside the unit circle], the coefficients ψ_j decline at an exponential rate as $j \to \infty$, possibly with oscillations.

Given the representation

$$\tilde{X}_t = \psi(B)u_t = \sum_{j=0}^{\infty} \psi_j u_{t-j}\,, \tag{7.70}$$

we can easily compute the autocovariances and autocorrelations of X_t :

$$\mathrm{Cov}(X_t, X_{t+k}) \;=\; \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}\,, \tag{7.71}$$

$$\mathrm{Corr}(X_t, X_{t+k}) \;=\; \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} \right) \Big/ \left(\sum_{j=0}^{\infty} \psi_j^2 \right). \tag{7.72}$$

However, this has the drawback of requiring one to compute limits of series.

7.6. Partial autocorrelations

The Yule-Walker equations allow one to determine the autocorrelations from the coefficients $\varphi_1, \ldots, \varphi_p$. In the same way we can determine $\varphi_1, \ldots, \varphi_p$ from the autocorrelations

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j), k = 1, 2, 3, ... \tag{7.73}$$

Taking into account the fact that $\rho(0) = 1$ and $\rho(-k) = \rho(k)$, we see that

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \ldots & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix} \tag{7.74}$$

or, equivalently,

$$R(p)\, \bar{\varphi}(p) = \bar{\rho}(p) \tag{7.75}$$

where

$$R(p) = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \ldots & 1 \end{bmatrix}, \tag{7.76}$$

$$\bar{\rho}(p) = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix}, \quad \bar{\varphi}(p) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix}. \tag{7.77}$$

Consider now the sequence of equations

$$R(k)\ \bar{\varphi}(k) = \bar{\rho}(k),\ k = 1, 2, 3, \ldots \tag{7.78}$$

where

$$R(k) = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(k-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(k-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(k-1) & \rho(k-2) & \rho(k-3) & \ldots & 1 \end{bmatrix}, \tag{7.79}$$

$$\bar{\rho}(k) = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{bmatrix}, \quad \bar{\varphi}(k) = \begin{bmatrix} \varphi(1|k) \\ \varphi(2|k) \\ \vdots \\ \varphi(k|k) \end{bmatrix},\ k = 1, 2, 3, \ldots \tag{7.80}$$

so that we can solve for $\bar{\varphi}(k)$:

$$\bar{\varphi}(k) = R(k)^{-1}\,\bar{\rho}(k). \tag{7.81}$$

[If $\sigma^2 > 0$, we can show that $R(k)^{-1}$ exists, $\forall k \geq 1$]. On using (7.75), we see easily that:

$$\varphi_k(k) = 0 \text{ for } k \geq p+1\,. \tag{7.82}$$

The coefficients φ_{kk} are called the lag- k *partial autocorrelations*.

In particular,

$$\varphi_1(\,|\,1) = \rho(1), \tag{7.83}$$

$$\varphi_2(2\,|\,2) = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2)-\rho(1)^2}{1-\rho(1)^2}, \tag{7.84}$$

$$\varphi_3(3\,|\,3) = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}. \tag{7.85}$$

The partial autocorrelations may be computed using the following recursive formulae:

$$\varphi(k+1\,|\,k+1) = \frac{\rho\,(k+1) - \sum_{j=1}^{k} \varphi(j\,|\,k)\,\rho\,(k+1-j)}{1 - \sum_{j=1}^{k} \varphi(j\,|\,k)\,\rho\,(j)}, \tag{7.86}$$

$$\varphi(j\,|\,k+1) = \varphi(j\,|\,k) - \varphi(k+1\,|\,k+1)\,\varphi(k+1-j\,|\,k)\,,\ \ j = 1, 2, \ldots, k. \tag{7.87}$$

Given $\rho(1), \ldots, \rho(k+1)$ and $\varphi_1(k), \ldots, \varphi_k(k)$, we can compute $\varphi_j(k+1)$, $j = 1, \ldots, k+1$. The expressions (7.86) - (7.87) are called the *Durbin-Levinson formulae*; see Durbin (1960) and Box and Jenkins (1976, pp. 82-84).

8. Mixed processes

Consider a process $\{X_t : t \in \mathbb{Z}\}$ which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{8.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$. Using operational notation, this can written

$$\varphi(B) X_t = \bar{\mu} + \theta(B) u_t. \tag{8.2}$$

8.1. Stationarity conditions

If the polynomial $\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p$ has all its roots outside the unit circle, the equation (8.1) has one and only one weakly stationary solution, which can be written:

$$X_t = \mu + \frac{\theta(B)}{\varphi(B)} u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \tag{8.3}$$

where

$$\mu = \bar{\mu}/\varphi(B) = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j), \tag{8.4}$$

$$\frac{\theta(B)}{\varphi(B)} := \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j. \tag{8.5}$$

The coefficients ψ_j are obtained by solving the equation

$$\varphi(B)\psi(B) = \theta(B). \tag{8.6}$$

In this case, we also have:

$$\mathbb{E}(X_{t-j} u_t) = 0, \forall j \geq 1. \tag{8.7}$$

The ψ_j coefficients may be computed in the following way (setting $\theta_0 = -1$):

$$\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right)\left(\sum_{j=0}^{\infty} \psi_j B^j\right) = 1 - \sum_{j=1}^{q} \theta_j B^j = -\sum_{j=1}^{q} \theta_j B^j \tag{8.8}$$

hence

$$\begin{aligned} \varphi(B)\psi_j &= -\theta_j \quad \text{for } j = 0, 1, \ldots, q \\ &= 0 \qquad\quad \text{for } j \geq q+1, \end{aligned} \tag{8.9}$$

where $\psi_j = 0$, for $j < 0$. Consequently,

$$\begin{aligned} \psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k} - \theta_j, \quad \text{for } j = 0, 1, \ldots, q \\ &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, \qquad\quad \text{for } j \geq q+1, \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \varphi_1 \psi_0 - \theta_1 = \varphi_1 - \theta_1, \\ \psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 - \theta_2 = \varphi_1 \psi_1 + \varphi_2 - \theta_2 = \varphi_1^2 - \varphi_1 \theta_1 + \varphi_2 - \theta_2, \\ &\vdots \\ \psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, \ j \geq q+1. \end{aligned} \tag{8.11}$$

The ψ_j coefficients behave like the autocorrelations of an AR(p) process, except for the initial coefficients $\psi_1, \ldots, \psi_q$.

8.2. Autocovariances and autocorrelations

Suppose:

$$\begin{aligned} &\text{a) the process } X_t \text{ is second-order stationary with } \sum_{j=1}^{p} \varphi_j \neq 1 \ ; \\ &\text{b) } \mathbb{E}(X_{t-j} u_t) = 0 \ , \forall j \geq 1 \ . \end{aligned} \tag{8.12}$$

By the stationarity assumption,

$$\mathbb{E}(X_t) = \mu, \forall t, \tag{8.13}$$

hence

$$\mu = \bar{\mu} + \sum_{j=1}^{p} \varphi_j \mu \tag{8.14}$$

and

$$\mathbb{E}(X_t) = \mu = \bar{\mu} / \left(1 - \sum_{j=1}^{p} \varphi_j \right). \tag{8.15}$$

The mean is the same as in the case of a pure AR(p) process. The MA(q) component of the model has no effect on the mean. Let us now rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{8.16}$$

where $\tilde{X}_t = X_t - \mu$. Consequently,

$$\tilde{X}_{t+k} = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t+k-j} + u_{t+k} - \sum_{j=1}^{q} \theta_j u_{t+k-j}\,, \tag{8.17}$$

$$\mathbb{E}(\tilde{X}_t \tilde{X}_{t+k}) = \sum_{j=1}^{p} \varphi_j \mathbb{E}(\tilde{X}_t \tilde{X}_{t+k-j}) + \mathbb{E}(\tilde{X}_t u_{t+k}) - \sum_{j=1}^{q} \theta_j \mathbb{E}(\tilde{X}_t u_{t+k-j})\,, \tag{8.18}$$

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) + \gamma_{xu}(k) - \sum_{j=1}^{q} \theta_j \gamma_{xu}(k-j)\,, \tag{8.19}$$

where

$$\begin{aligned} \gamma_{xu}(k) &= \mathbb{E}(\tilde{X}_t u_{t+k}) = 0\,, \quad \text{if } k \geq 1\,, \\ &\neq 0\,, \qquad\qquad \text{if } k \leq 0\,, \\ \gamma_{xu}(0) &= \mathbb{E}(\tilde{X}_t u_t) = \sigma^2. \end{aligned} \tag{8.20}$$

For $k \geq q+1$,

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j)\,, \tag{8.21}$$

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j)\,. \tag{8.22}$$

The variance is given by

$$\gamma(0) = \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)\,, \tag{8.23}$$

hence

$$\gamma(0) = \left[\sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)\right] / \left[1 - \sum_{j=1}^{p} \varphi_j \rho(j)\right]\,. \tag{8.24}$$

In operational notation, the autocovariances satisfy the equation

$$\varphi(B)\gamma(k) = \theta(B)\gamma_{xu}(k)\ ,\ k \geq 0, \tag{8.25}$$

where $\gamma(-k) = \gamma(k)$, $B^j\gamma(k) := \gamma(k-j)$ and $B^j\gamma_{xu}(k) := \gamma_{xu}(k-j)$. In particular,

$$\varphi(B)\gamma(k) \;=\; 0\ ,\ \text{for } k \geq q+1, \tag{8.26}$$

$$\varphi(B)\rho(k) \;=\; 0\ ,\ \text{for } k \geq q+1. \tag{8.27}$$

To compute the autocovariances, we can solve the equations (8.19) for $k = 0, 1, \ldots, p$, and then apply (8.21). The autocorrelations of an process ARMA(p, q) process behave like those of an AR(p) process, except that initial values are modified.

Example 8.1 Consider the ARMA$(1,1)$ model:

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t - \theta_1 u_{t-1} \ , |\varphi_1| < 1 \tag{8.28}$$

$$\tilde{X}_t - \varphi_1 \, \tilde{X}_{t-1} = u_t - \theta_1 u_{t-1} \tag{8.29}$$

where $\tilde{X}_t = X_t - \mu$. We have

$$\begin{aligned} \gamma(0) &= \varphi_1 \gamma(1) + \gamma_{xu}(0) - \theta_1 \gamma_{xu}(-1), & (8.30) \\ \gamma(1) &= \varphi_1 \gamma(0) + \gamma_{xu}(1) - \theta_1 \gamma_{xu}(0) & (8.31) \end{aligned}$$

and

$$\begin{aligned} \gamma_{xu}(1) &= 0, & (8.32) \\ \gamma_{xu}(0) &= \sigma^2, & (8.33) \\ \gamma_{xu}(-1) &= \mathbb{E}(\tilde{X}_t u_{t-1}) = \varphi_1 \mathbb{E}(\tilde{X}_{t-1} u_{t-1}) + \mathbb{E}(u_t u_{t-1}) - \theta_1 \mathbb{E}(u_{t-1}^2) \\ &= \varphi_1 \gamma_{xu}(0) - \theta_1 \sigma^2 = (\varphi_1 - \theta_1)\sigma^2 & (8.34) \end{aligned}$$

Thus,

$$\begin{aligned} \gamma(0) &= \varphi_1 \gamma(1) + \sigma^2 - \theta_1(\varphi_1 - \theta_1)\sigma^2 \\ &= \varphi_1 \gamma(1) + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2, & (8.35) \end{aligned}$$

$$\begin{aligned} \gamma(1) &= \varphi_1 \gamma(0) - \theta_1 \sigma^2 \\ &= \varphi_1 \{\varphi_1 \gamma(1) + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2\} - \theta_1 \sigma^2 \ , & (8.36) \end{aligned}$$

hence

$$\begin{aligned}
\gamma(1) &= \{\varphi_1[1-\theta_1(\varphi_1-\theta_1)]-\theta_1\}\sigma^2/(1-\varphi_1^2) \\
&= \{\varphi_1-\theta_1\varphi_1^2+\varphi_1\theta_1^2-\theta_1\}\sigma^2/(1-\varphi_1^2) \\
&= (1-\theta_1\varphi_1)(\varphi_1-\theta_1)\sigma^2/(1-\varphi_1^2)\,. \qquad (8.37)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\gamma(0) &= \varphi_1\gamma(1)+[1-\theta_1(\varphi_1-\theta_1)]\sigma^2 \\
&= \varphi_1\frac{(1-\theta_1\varphi_1)\,(\varphi_1-\theta_1)\,\sigma^2}{1-\varphi_1^2}+[1-\theta_1(\varphi_1-\theta_1)]\sigma^2 \\
&= \frac{\sigma^2}{1-\varphi_1^2}\{\varphi_1(1-\theta_1\varphi_1)(\varphi_1-\theta_1)+(1-\varphi_1^2)[1-\theta_1(\varphi_1-\theta_1)]\} \\
&= \frac{\sigma^2}{1-\varphi_1^2}\{\varphi_1^2-\theta_1\varphi_1^3+\varphi_1^2\theta_1^2-\varphi_1\theta_1+1-\varphi_1^2-\theta_1\varphi_1+\theta_1\varphi_1^3+\theta_1^2-\varphi_1^2\theta_1^2\} \\
&= \frac{\sigma^2}{1-\varphi_1^2}\{1-2\,\varphi_1\theta_1+\theta_1^2\}\,. \qquad (8.38)
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma(0) &= (1-2\,\varphi_1\theta_1+\theta_1^2)\sigma^2/(1-\varphi_1^2)\,, && (8.39) \\
\gamma(1) &= (1-\theta_1\varphi_1)(\varphi_1-\theta_1)\sigma^2/(1-\varphi_1^2)\,, && (8.40) \\
\gamma(k) &= \varphi_1\gamma(k-1),\ \text{for } k\geq 2\,. && (8.41)
\end{aligned}$$

9. Invertibility

A second-order stationary AR(p) process in MA(∞) form. Similarly, any second-order stationary ARMA(p,q) process can also be expressed as MA(∞) process. By analogy, it is natural to ask the question: can an MA(q) or ARMA(p,q) process be represented in a autoregressive form?

Consider the $MA(1)$ process

$$X_t = u_t - \theta_1 u_{t-1}, t \in \mathbb{Z}\,, \tag{9.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$ and $\sigma^2 > 0$. We see easily that

$$\begin{aligned} u_t &= X_t + \theta_1 u_{t-1} \\ &= X_t + \theta_1 (X_{t-1} + \theta_1 u_{t-2}) \\ &= X_t + \theta_1 X_{t-1} + \theta_1^2 u_{t-2} \\ &= \sum_{j=0}^{n} \theta_1^j X_{t-j} + \theta_1^{n+1} u_{t-n-1} \end{aligned} \tag{9.2}$$

and

$$\mathbb{E}\left[\left(\sum_{j=0}^{n} \theta_1^j X_{t-j} - u_t\right)^2\right] = \mathbb{E}\left[\left(\theta_1^{n+1} u_{t-n-1}\right)^2\right] = \theta_1^{2(n+1)} \sigma^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{9.3}$$

provided $|\theta_1| < 1$. Consequently, the series $\sum\limits_{j=0}^{n} \theta_1^j X_{t-j}$ converges in $q.m.$ to u_t if $|\theta_1| < 1$. In other words,

when $|\theta_1| < 1$, we can write

$$\sum_{j=0}^{\infty} \theta_1^j X_{t-j} = u_t, t \in \mathbb{Z}\,, \tag{9.4}$$

or

$$(1-\theta_1 B)^{-1} X_t = u_t, t \in \mathbb{Z} \tag{9.5}$$

where $(1-\theta_1 B)^{-1} = \sum_{j=0}^{\infty} \theta_1^j B^j$. The condition $|\theta_1| < 1$ is equivalent to having the roots of the equation $1-\theta_1 z = 0$ outside the unit circle. If $\theta_1 = 1$,

$$X_t = u_t - u_{t-1} \tag{9.6}$$

and the series

$$(1-\theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} \theta_1^j X_{t-j} = \sum_{j=0}^{\infty} X_{t-j} \tag{9.7}$$

does not converge, for $\mathbb{E}(X_{t-j}^2)$ does not converge to 0 as $j \to \infty$. Similarly, if $\theta_1 = -1$,

$$X_t = u_t + u_{t-1} \tag{9.8}$$

and the series

$$(1-\theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} (-1)^j X_{t-j} \tag{9.9}$$

does not converge either. These models are not invertible.

Theorem 9.1 INVERTIBILITY CONDITION FOR A MA PROCESS. *Let $\{X_t : t \in \mathbb{Z})$ be a second-order stationary process such that*

$$X_t = \mu + \theta(B)u_t \tag{9.10}$$

where $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. Then the process X_t satisfies an equation of the form

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\mu} + u_t \tag{9.11}$$

iff the roots of the polynomial $\theta(z)$ are outside the unit circle. Further, when the representation (9.11) exists, we have:

$$\bar{\phi}(B) = \theta(B)^{-1}, \bar{\mu} = \theta(B)^{-1}\mu = \mu \,/\, \left(1 - \sum_{j=1}^{q} \theta_j\right). \tag{9.12}$$

Corollary 9.2 INVERTIBILITY CONDITON FOR AN ARMA PROCESS. *Let $\{X_t : t \in \mathbb{Z}\}$ be a second-order stationary ARMA process that satisfies the equation*

$$\varphi(B)X_t = \bar{\mu} + \theta(B)u_t \tag{9.13}$$

where $\varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. Then the process X_t satisfies an equation of the form

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\bar{\mu}} + u_t \tag{9.14}$$

iff the roots du polynomial $\theta(z)$ are outside the unit circle. Further, when the representation (9.14) exists, we have:

$$\bar{\phi}(B) = \theta(B)^{-1}\varphi(B), \quad \bar{\bar{\mu}} = \theta(B)^{-1}\bar{\mu} = \mu \,/\, \left(1 - \sum_{j=1}^{q} \theta_j\right). \tag{9.15}$$

10. Wold representation

We have seen that all second-order ARMA processes can be written in a causal MA(∞) form. This property indeed holds for all second-order stationary processes.

Theorem 10.1 WOLD REPRESENTATION OF WEAKLY STATIONARY PROCESSES. *Let $\{X_t,\ t \in \mathbb{Z}\}$ be a second-order stationary process such that $\mathbb{E}(X_t) = \mu$. Then X_t can be written in the form*

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} + v_t \tag{10.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0,\ \sigma^2)$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, $\mathbb{E}(u_t X_{t-j}) = 0,\ \forall j \geq 1$, and $\{v_t : t \in \mathbb{Z}\}$ is a deterministic process such that $\mathbb{E}(v_t) = 0$ and $\mathbb{E}(u_s v_t) = 0,\ \forall s, t$. Further, if $\sigma^2 > 0$, the sequences $\{\psi_j\}$ and $\{u_t\}$ are unique, and

$$u_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t-1}, \tilde{X}_{t-2}, \ldots) \tag{10.2}$$

where $\tilde{X}_t = X_t - \mu$.

PROOF. See Anderson (1971, Section 7.6.3, pp. 420-421) and Hannan (1970, Chapter III, Section 2, Theorem 2, pp. 136-137). □

If $\mathbb{E}(u_t^2) > 0$ in Wold representation, we say the process X_t is *regular*. v_t is called the deterministic *component of* the process while $\sum_{j=0}^{\infty} \psi_j u_{t-j}$ is its *indeterministic component*. When $v_t = 0, \forall t$, the process X_t is said to be *strictly indeterministic*.

Corollary 10.2 FORWARD WOLD REPRESENTATION OF WEAKLY STATIONARY PROCESSES. *Let $\{X_t : t \in \mathbb{Z}\}$ be second-order a stationary process such that $\mathbb{E}(X_t) = \mu$. Then X_t can be written in the form*

$$X_t = \mu + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j} + \bar{v}_t \tag{10.3}$$

where $\{\bar{u}_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \bar{\sigma}^2)$, $\sum_{j=0}^{\infty} \bar{\psi}_j^2 < \infty$, $\mathbb{E}(\bar{u}_t X_{t+j}) = 0$, $\forall j \geq 1$, and $\{\bar{v}_t : t \in \mathbb{Z}\}$ is a deterministic (with respect to $\bar{v}_{t+1}$, $\bar{v}_{t+2}$, ...) such that $\mathbb{E}(\bar{v}_t) = 0$ and $\mathbb{E}(\bar{u}_s \bar{v}_t) = 0$, $\forall s, t$. Further, if $\bar{\sigma}^2 > 0$, the sequences $\{\bar{\psi}_j\}$ and $\{\bar{u}_t\}$ are uniquely defined, and

$$\bar{u}_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t+1}, \tilde{X}_{t+2}, \ldots) \tag{10.4}$$

where $\tilde{X}_t = X_t - \mu$.

PROOF. The result follows on applying Wold theorem to the process $Y_t := X_{-t}$ which is also second-order stationary. □

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