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September 2011
Compiled: February 10, 2014

ECONOMETRIC THEORY
EXERCISES 9

MAXIMUM LIKELIHOOD METHOD

Reference: Gouriéroux and Monfort (1995, Chapter)

1. Consider the density function

$$\ell(y_1, \ldots, y_n; \theta) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - \theta)^2\right\}.$$

   (a) By modifying this density on a set of zero Lebesgue measure, show it is possible to make the maximum likelihood estimator equal to $\sum_{i=1}^{n} y_i^4$.

   (b) Is it possible to preclude this type of manipulation? If so, how?

2. In maximum likelihood problems, show that:

   (a) the maximum likelihood estimator may not exist;

   (b) multiple maximum likelihood estimators may exist.

3. Let $\ell(y; \theta)$, $\theta \in \Theta$, be a likelihood function such that

   (a) Θ is a convex set
   and

   (b) $\log[\ell(y; \theta)]$ is a strictly concave function of θ.

   Show that the maximum likelihood estimator of θ is unique (if it does exist).

4. What happens to the maximum likelihood estimator when the model is reparameterized? Justify your answer.

5. Let $(\mathscr{Y}, \mathscr{P})$ where $\mathscr{P} = (P_\theta = \ell(y; \theta) \cdot \mu,\ \theta \in \Theta)$, a dominated parametric model, and $S(y)$ a sufficient statistic for θ.

(a) If $\lambda = g(\theta)$ is a one-to-one function (bijection) of θ and if the maximum likelihood estimator $\hat{\theta}(y)$ of θ is unique, how are the maximum likleihood estimators of λ and θ related? Justify your answer.

(b) What happens when the maximum likelihood estimator of θ is not unique?

(c) Is the estimator $\hat{\theta}(y)$ a function of $S(y)$? Justify your answer.

6. Consider the equilibrium model:

$$\begin{aligned} q_t &= a p_t + b + u_t\,, \\ S_t &= \alpha p_t + \beta x_t + v_t\,, \\ q_t &= S_t\,, \end{aligned}$$

where q_t is the quantity demanded, p_t is the price, S_t is quantity supplied, x_t is an exogenous variable, and the vectors $(u_t, v_t)'$, $t = 1, \ldots, n$ are independent with the same distribution

$$N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}\right].$$

(a) Find the reduced form of this model.

(b) How are the parameters of this reduced form related to the structural form? Is this model underidentified, just identified, or overidentified ?

(c) Find the maximum likelihood estimators of the reduced-form coefficients.

(d) Find the maximum likelihood estimators of the structural-form coefficients.

7. Give regularity conditions under which a sequence of maximum likelihood estimators converges almost surely to the true parameter value.

8. Consider the following assumptions:

H1: the variables $Y_1, \ldots, Y_n$ are independent and follow the same distribution with density $f(y;\,\theta)$, $\theta \in \Theta \subseteq \mathbb{R}^p$;

H2: the interior of Θ is non-empty, and θ_0 belongs to the interior of Θ;

H3: the true unknown value θ_0 is identifiable;

H4: the log-likelihood

$$L_n(y;\,\theta) = \sum_{i=1}^{n} \log\left[f(y_i;\,\theta)\right] \text{ is continuous in } \theta\,;$$

H5: $\mathsf{E}_{\theta_0}\left[\log f(Y_i;\,\theta)\right]$ is finite;

H6: the log-likelihood is such that $\frac{1}{n} L_n(y;\, \theta)$ converges almost surely to $\mathsf{E}_{\theta_0}[\log(Y_i;\, \theta)]$ uniformly in $\theta \in \Theta$;

H7: the log-likelihood is twice continuously differentiable in open neighborhood of θ_0;

H8: $I_1(\theta_0) = \mathsf{E}_{\theta_0}\left[-\frac{\partial^2 \log f(Y;\, \theta)}{\partial\theta\, \partial\theta'}\right]$ is finite and invertible.

If $\hat{\theta}_n$ is consistent sequence of local maxima, show that the asymptotic distribution of $\sqrt{n}\big(\hat{\theta}_n - \theta_0\big)$ is $N\big[0,\, I_1(\theta_0)^{-1}\big]$.

9. Let $Y_1, \ldots, Y_n$ be a sample of independent identically distributed random variables with distribution $N\left[\mu,\, \sigma^2\right]$ where $\mu \neq 0$ and $\sigma > 0$. Find the asymptotic distribution of the maximum likelihood estimator of $\gamma = 1/\mu$.

10. Let $Y_1, \ldots, Y_n$ be a random sample of independent random variables from the exponential distribution with density:

$$f(y;\, \theta) = e^{-(y-\theta)} 1_{y \geq \theta}\,.$$

   (a) Which regularity condition is not satisfied in this problem?

   (b) How does $\sqrt{n}\big(\hat{\theta}_n - \theta_0\big)$ behave for large n?

   (c) Provide the asymptotic distribution of $\hat{\theta}_n$.

References

GOURIÉROUX, C., AND A. MONFORT (1995): *Statistics and Econometric Models, Volumes One and Two*. Cambridge University Press, Cambridge, U.K., Translated by Quang Vuong.