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ADVANCED ECONOMETRIC THEORY
EXERCISES 5
UNBIASED ESTIMATION

1. **Identification and unbiased estimation**. Demonstrate the following relationship between identifiability and unbiased estimation:
   if a function $g(\theta)$ of a parameter θ is not identifiable, then there is no unbiased estimator of $g(\theta)$.

2. **Regular model**. When is a dominated parametric model *regular*?

3. **Fréchet-Darmois-Cramér-Rao inequality**

   (a) State the Fréchet-Darmois-Cramér-Rao inequality.

   (b) Prove the Fréchet-Darmois-Cramér-Rao inequality.

4. **Lehmann-Scheffé theorem**. State and demonstrate the Lehmann-Scheffé theorem.

5. **Properties of best unbiased estimators**. Let $T^*(Y)$ be an optimal unbiased estimator of $g(\theta)$ and let $T(Y)$ be any other unbiased estimator of $g(\theta)$. [The risk function is (matrix) quadratic risk.]

   (a) Show that $T^*(Y)$ and $T(Y) - T^*(Y)$ are uncorrelated.

   (b) Show that the best optimal estimator is unique.

6. **Least squares as best unbiased estimators**. Consider the classical linear model

   $$y = X\beta + u$$

   where X is a fixed matrix of dimension $n \times k$ such that $1 \leq \text{rang}(X) = k < n$ and $u \sim N[0, \sigma^2 I_n]$.

   (a) Show that $\hat{\beta} = (X'X)^{-1} X'y$ and $s^2 = \hat{u}'\hat{u}/(n-k)$, where $\hat{u} = y - X\hat{\beta}$, are sufficient statistics for the parameter vector $(\beta', \sigma^2)'$.

   (b) Show that $\hat{\beta}$ and s^2 are optimal among all unbiased estimators of β and σ^2 (according to matrix quadratic risk).