Jean-Marie Dufour
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ADVANCED ECONOMETRIC THEORY
EXERCICES 11

M-ESTIMATORS

1. (a) Define the notion of M-estimator.
   (b) Explain the difference between “M-estimators” and “maximum likelihood estimators”.

   (a) Give regularity conditions under which an M-estimator converges almost surely to a constant.
   (b) To what this constant corresponds?
   (c) Give regularity conditions under which the M-estimator has a normal asymptotic distribution, and derive this distribution. Provide the asymptotic covariance matrix of the M-estimator.

2. Is is possible to establish the asymptotic distribution of the maximum likelihood estimator from the one of M-estimators? If so, explain how.

3. (a) Define what is a quasi-generalized M-estimator.
   (b) Give a condition under which the distribution of a quasi-generalized M-estimator does not depend on the asymptotic distribution of the first-step estimator $(\tilde{c}_n)$.
   (c) What is the form of the covariance matrix of quasi-generalized M-estimators?

4. Consider the nonlinear regression model:

$$
\begin{aligned}
& Y_i = h(X_i, \beta_0) + u_i, \ \beta_0 \in \mathscr{B} \\
& \mathsf{E}(u_i \mid X_1, \dots, X_n) = 0 \\
& \mathsf{E}\left(u_i^2 \mid X_1, \dots, X_n\right) = \omega^2(X_i, \beta_0) > 0, \quad i = 1, \dots, n
\end{aligned}
$$

   where

   H1: the pairs (Y_i, X_i), $i = 1, \dots, n$ are independent and identically distributed;

H2: $\mathscr{B}$ is a compact set;

H3: $h(X, \beta)$ is a continuous function of β and

$$\mathsf{E}\left[(Y_i - h(X_i, \beta))^2\right] < \infty, \ \forall \beta \in \mathscr{B};$$

H4: $\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - h(X_i, \beta)\right)^2$ converges almost surely and uniformly on $\mathscr{B}$ to $\mathsf{E}\left[\left(Y_i - h(X_i, \beta)\right)^2\right]$.

(a) When is the parameter β first-order identified? When is it second-order identified?

(b) If we suppose that β is first-order identified, show that the estimator $\hat{\beta}_n$ obtained by minimizing $\sum_{i=1}^{n}\left(Y_i - h(X_i, \beta)\right)^2$ (nonlinear least squares estimator) is consistent.

(c) If we suppose that β is first-order identified, give regularity conditions under which the asymptotic distribution of $\sqrt{n}\left(\hat{\beta}_n - \beta_0\right)$ is normal. Give the asymptotic covariance matrix of $\sqrt{n}\left(\hat{\beta}_n - \beta_0\right)$.

(d) Find an estimator of β whose asymptotic variance cannot be worse than the one of $\hat{\beta}_n$.

To answer 4b and 4c, you can use the general theory of *M*-estimators.

5. Exercise 8.3 in Gouriéroux and Monfort (1995, chap. 8).

6. Exercise 8.4 in Gouriéroux and Monfort (1995, chap. 8).

References

GOURIÉROUX, C., AND A. MONFORT (1995): *Statistics and Econometric Models, Volumes One and Two*. Cambridge University Press, Cambridge, U.K., Translated by Quang Vuong.