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TIME SERIES ANALYSIS
EXERCISES
STOCHASTIC PROCESSES 2

1. Discuss the convergence conditions for the series

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$$

where $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$. In particular, give sufficient conditions under which:

   (a) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in mean of order 2;

   (b) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in mean of order $r > 0$;

   (c) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges almost surely;

   (d) $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ converges in probability.

2. Consider a $MA(1)$ model:

$$X_t = \overline{\mu} + u_t - \theta u_{t-1}\,,\; t \in \mathbb{Z}$$

where $u_t \sim WN(0, \sigma^2)$ and $\sigma^2 > 0$.

   (a) Prove that the first autocorrelation of this model cannot be greater than 0.5 in absolute value.

   (b) Find the values of the model parameters for which this upper bound is attained.

3. Let $\{x_t : t \in \mathbb{Z}\}$ an MA(q) process. For $q = 3, 4, 5, 6$, check whether the following inequalities are correct:

   (a) $|\rho(1)| \leq 0.75$;

   (b) $|\rho(2)| \leq 0.90$;

(c) $|\rho(3)| \leq 0.90$;

(d) $|\rho(4)| \leq 0.90$;

(e) $|\rho(5)| \leq 0.90$;

(f) $|\rho(6)| \leq 0.90$.

4. Consider the following models:

$$
\begin{array}{ll}
(1) & X_t = 0.5\, X_{t-1} + u_t\,, \\
(2) & X_t = 10 - 0.75\, X_{t-1} + u_t\,, \\
(3) & X_t = 10 + 0.7\, X_{t-1} - 0.2\, X_{t-2} + u_t\,, \\
(4) & X_t = 10 + u_t - 0.75\, u_{t-1} + 0.125\, u_{t-2}, \\
(5) & X_t = 0.5\, X_{t-1} + u_t - 0.25\, u_{t-1}, \\
(6) & X_t = 0.5\, X_{t-1} + u_t - 0.5\, u_{t-1},
\end{array}
$$

where $\{u_t : t \in \mathbb{Z}\}$ is an $i.i.d.$ $N(0,1)$ sequence. For each one of these models, answer the following questions.

(a) Is this model stationary? Why?

(b) Is this model invertible? Why?

(c) Compute:

i. $E(X_t)$;

ii. $\gamma(k)$, $k = 1, \ldots, 8$;

iii. $\rho(k)$, $k = 1, 2, \ldots, 8$.

(d) Graph $\rho(k)$, $k = 1, 2, \ldots, 8$.

(e) Find the coefficients of u_t, u_{t-1}, u_{t-2}, u_{t-3} and u_{t-4} in the moving average representation of X_t.

(f) Find the autocovariance generating function of X_t.

(g) Find and graph the spectral density of X_t.

(h) Compute the first four partial autocorrelations of X_t.