ARIMA model validation *

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1. Problem

$$X_t \sim ARIMA\,(p, d, q)$$
$$\varphi\,(B)\,\nabla^d X_t = \varphi_0 + \Theta\,(B)\,a_t\ .$$

After estimation of the parameters. we expect that residuals $\hat{a}_t$ be approximately a white noise.

The criterion of success for ARIMA model is: to reduce a time series to the white noise structure.

Let us examine how we can test whether a series $a, \ldots, a_N$ is a white noise.

2. Correlogram of a white noise

Let

$$a_1, ..., a_N \sim BB\left(0, \sigma_a^2\right)\ . \tag{2.1}$$

Then

$$r_k\,(a) = \sum_{t=1}^{N-k} a_t a_{t+k} \,/\, \sum_{t=1}^{N} a_t^2 \tag{2.2}$$

is an estimator of $E\,(a_t a_{t+k})\,/E(a_t^2)$.

For N large,

$$r_k(a) \sim N\left[0, \frac{1}{N}\right] \tag{2.3}$$
$$\frac{r_k\,(a)}{1/\sqrt{N}} \sim N\,[0, 1]\ . \tag{2.4}$$

Furthermore, we can show that $r_k\,(a)\,, k = 1, ..., K$, where $K < N$, are independent. Hence :

$$Q\,(r) = \sum_{k=1}^{K}\left[\frac{r_k(a)}{1/\sqrt{N}}\right]^2 = N\sum_{k=1}^{K} r_k\,(a)^2 \sim \chi^2\,(K)\ .$$

We can test whether $a_1, ..., a_N$ constitute a white noise.

3. Correlogram of residuals

Instead of $a_1, ..., a_N$, we have $\hat{a}_1, ..., \hat{a}_N$. We wish to test

$$H_0 : \text{an ARIMA}\,(p, d, q) \text{ is adequate.} \tag{3.1}$$

Let us examine the autocorrelations:

$$r_k\,(\hat{a})\ , \quad k = 1,\ \dots\ ,\ K.$$

For N large,

$$\sqrt{N}\, r_k\,(\hat{a}) \overset{a}{\sim} N\,[0,\ 1]\ , \quad k = 1,\ \dots\ ,\ K.$$

but they are not independent.

However, one can show [Box and Pierce (1970)] that

$$\hat{r} \simeq (I - D)\, r$$

where

$$r = \begin{pmatrix} r_1\,(a) \\ \vdots \\ r_K\,(a) \end{pmatrix}\ , \quad \hat{r} = \begin{pmatrix} r_1\,(\hat{a}) \\ \vdots \\ r_K\,(\hat{a}) \end{pmatrix} \tag{3.2}$$

and

$$I_K - D \text{ is an idempotent matrix of rank } K - \ell,\ \ell = p + q. \tag{3.3}$$

Thus

$$\sqrt{N}\, r \overset{a}{\sim} N_K\,(0,\ I_K) \tag{3.4}$$

$$\sqrt{N}\, \hat{r} \simeq (I_K - D)\sqrt{N}\, r \overset{a}{\sim} N_K\,(0,\ I_K - D) \tag{3.5}$$

$$Q\,(\hat{r}) = N \sum_{k=1}^{K} r_k\,(\hat{a})^2 \sim \chi^2\,(K - \ell)$$

$p + q$ does not include the constant.

4. Ljung-Box statistic

For relatively short series, approximating the distribution of Q by a $\chi^2\,(K - \ell)$ distribution can yield very unreliable results [see Davies, Triggs and Newbold (1977)]. In particular,

$$E\,(Q) < E\left[\chi^2\,(K - \ell)\right]\ . \tag{4.1}$$

Ljung and Box (1978) have proposed a modification which improves the approximation. Consider first the case of white noise:

$$Var\left[r_k\left(a\right)\right] = \frac{N-k}{N\left(N+2\right)}\,, \quad k = 1, 2, \ldots, K\,,$$

$$\frac{r_k\left(a\right)}{\sqrt{\frac{N-k}{N(N+2)}}} \sim N\left(0,\,1\right) \tag{4.2}$$

where

$$\frac{N-k}{N\left(N+2\right)} < \frac{1}{N}\,. \tag{4.3}$$

$$\tilde{Q}\left(r\right) = \sum_{k=1}^{K}\left[\frac{r_k\left(a\right)}{\sqrt{\frac{N-k}{N(N+2)}}}\right]^2 = N\left(N+2\right)\sum_{k=1}^{K}\frac{r_k\left(a\right)^2}{N-k}\,.$$

With estimated residuals, we use:

$$\tilde{Q}\left(\hat{r}\right) = N\left(N+2\right)\sum_{k=1}^{K}\left(N-k\right)^{-1} r_k\left(\hat{a}\right)^2 \sim \chi^2\left(K-\ell\right).$$

This statistic is called the Ljung-Box statistic

References

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