Prediction and regression *

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Contents

List of Definitions, Assumptions, Propositions and Theorems **1**

1. Optimal mean square prediction **1**

2. Properties of conditional expectations **2**

3. Linear regression **3**

4. Bibliographic notes **6**

List of Definitions, Assumptions, Propositions and Theorems

2.1 Proposition : Linearity . . . 2
2.2 Proposition : Positivity . . . 2
2.3 Proposition : Monotonicity . . . 2
2.4 Proposition : Invariance . . . 2
2.5 Proposition : Orthogonality . . . 2
2.6 Proposition : Iterated conditionings law . . . 2
2.7 Proposition : Mean square optimality . . . 3
2.8 Proposition : Characterization of optimality by orthogonality . . . 3
2.9 Definition : Conditional covariance . . . 3
2.10 Proposition : Variance decomposition . . . 3

1. Optimal mean square prediction

Let $Y,\ X_1, \ldots, X_k$ be real random variables in L^2, and $X = (X_1, \ldots, X_k)'$. We wish to find a function

$$g(X) = g(X_1, \ldots, X_k)$$

such that

$$\mathsf{E}\left([Y - g(X)]^2 \right) \text{ is minimal.}$$

Given the mean square criterion, we also restrict $g(X)$ to be in L^2 :

$$\mathsf{E}\left[g(X)^2 \right] < \infty.$$

Then it is easy to see that the optimal solution to this problem is

$$g(X) = M(X)$$

where

$$M(X) = \mathsf{E}(Y \mid X)\ .$$

In general, $M(X)$ is a nonlinear function of X. The optimality of $M(X)$ can easily be shown on observing that :

$$\begin{aligned}
\mathsf{E}\left\{ [Y - g(X)]^2 \right\} &= \mathsf{E}\left\{ [Y - \mathsf{E}(Y \mid X) + \mathsf{E}(Y \mid X) - g(X)]^2 \right\} \\
&= \mathsf{E}\Big\{ [Y - \mathsf{E}(Y \mid X)]^2 + [\mathsf{E}(Y \mid X) - g(X)]^2 \\
&\quad + 2\,[Y - \mathsf{E}(Y \mid X)]\,[\mathsf{E}(Y \mid X) - g(X)] \Big\} \\
&= \mathsf{E}\left\{ [Y - \mathsf{E}(Y \mid X)]^2 \right\} + \mathsf{E}\left\{ [\mathsf{E}(Y \mid X) - g(X)]^2 \right\} \\
&\quad + 2E\left\{ [\mathsf{E}(Y \mid X) - g(X)]\,\mathsf{E}[Y - \mathsf{E}(Y \mid X) \mid X] \right\} \\
&= \mathsf{E}\left\{ [Y - \mathsf{E}(Y \mid X)]^2 \right\} + \mathsf{E}\left\{ [\mathsf{E}(Y \mid X) - g(X)]^2 \right\}
\end{aligned}$$

from which it follows that the optimal solution is

$$g(X) = \mathsf{E}(Y \mid X)\ .$$

The set of random variables

$$M_0 = \left\{ Z : Z = g(X) \text{ is a random variable and } \mathsf{E}\left(Z^2\right) < \infty \right\}$$

is a closed subspace of L^2. $M(X) = \mathsf{E}(Y \mid X)$ can be interpreted as the projection of Y on M_0 :

$$\mathsf{E}(Y \mid X) = P_{M_0} Y.$$

2. Properties of conditional expectations

Let

$$\begin{aligned} Y &= (Y_1, \ldots, Y_q)' , \\ Z &= (Z_1, \ldots, Z_q)' , \\ X &= (X_1, \ldots, X_k) \end{aligned}$$

be random vectors whose components are all in L^2. By definition,

$$\mathsf{E}(Y \mid X) = \begin{bmatrix} \mathsf{E}(Y_1 \mid X) \\ \mathsf{E}(Y_2 \mid X) \\ \vdots \\ \mathsf{E}(Y_q \mid X) \end{bmatrix}$$

and similarly for $\mathsf{E}(Z \mid X)$.

Let $L^2(X)$ be the set of random variables W such that $W = g(X)$ and $\mathsf{E}(W^2) < \infty$.

2.1 Proposition LINEARITY. *Let A an $m \times q$ fixed matrix and b an $m \times 1$ fixed vector. Then*

$$\begin{aligned} \mathsf{E}(AY + b \mid X) &= AE(Y \mid X) + b, \\ \mathsf{E}(Y + Z \mid X) &= \mathsf{E}(Y \mid X) + \mathsf{E}(Z \mid X) . \end{aligned}$$

2.2 Proposition POSITIVITY. *If $Y_i \geq 0$, for $i = 1, \ldots, q$, then*

$$\mathsf{E}(Y_i \mid X) \geq 0 , \quad \text{for} \quad i = 1, \ldots, q .$$

2.3 Proposition MONOTONICITY. *If $Y_i \geq Z_i$, for $i = 1, \ldots, q$, then*

$$\mathsf{E}(Y_i \mid X) \geq \mathsf{E}(Z_i \mid X) , \quad \text{for} \quad i = 1, \ldots, q .$$

2.4 Proposition INVARIANCE.

$$\begin{aligned} \mathsf{E}(Y \mid X) = Y \quad &\Leftrightarrow \quad Y \text{ is a function of } X \\ &\Leftrightarrow \quad \text{there is a function } g(x) \text{ such that } Y = g(X) \\ &\qquad \text{with probability } 1. \end{aligned}$$

2.5 Proposition ORTHOGONALITY. *If $g_1(X) \in L^2$ and $g_2(Y) \in L^2$, then*

$$\mathsf{E}\{g_1(X)[g_2(Y) - \mathsf{E}(g_2(Y) \mid X)]\} = 0 .$$

2.6 Proposition ITERATED CONDITIONINGS LAW. *If W is a random vector such that*

$$L^2(W) \subseteq L^2(X) ,$$

then

$$\begin{aligned} \mathsf{E}[\mathsf{E}(Y \mid X) \mid W] &= \mathsf{E}[\mathsf{E}(Y \mid W) \mid X] \\ &= \mathsf{E}(Y \mid W) . \end{aligned}$$

2.7 Proposition MEAN SQUARE OPTIMALITY.

$$\mathsf{E}\left[(Y_i - \mathsf{E}(Y_i \mid X))^2\right] = \min_{g_i(X) \in L^2(X)} \mathsf{E}\left[(Y_i - g_i(X))^2\right], \ i = 1, \ldots, q .$$

2.8 Proposition CHARACTERIZATION OF OPTIMALITY BY ORTHOGONALITY. *For any* $i = 1, \ldots, q$,

$$h_i(X) = \mathsf{E}(Y_i \mid X) \Leftrightarrow \mathsf{E}[g(X)(Y_i - h_i(X))] = 0 , \ \forall g(X) \in L^2(X) .$$

2.9 Definition CONDITIONAL COVARIANCE. *The conditional covariance matrix of* Y *given* X *is the matrix*

$$\mathsf{V}(Y \mid X) = \mathsf{E}\left[(Y - \mathsf{E}(Y \mid X))(Y - \mathsf{E}(Y \mid X))' \mid X\right] .$$

If we define

$$\varepsilon(X) = Y - \mathsf{E}(Y \mid X) ,$$

we see easily that

$$\mathsf{V}[\varepsilon(X)] = \mathsf{E}[\mathsf{V}(Y \mid X)] .$$

We can then write

$$Y = \mathsf{E}(Y \mid X) + \varepsilon(X)$$

where $\mathsf{E}(Y \mid X)$ and $\varepsilon(X)$ are uncorrelated.

2.10 Proposition VARIANCE DECOMPOSITION.

$$\begin{aligned} \mathsf{V}(Y) &= \mathsf{V}[\mathsf{E}(Y \mid X)] + \mathsf{V}[\varepsilon(X)] \\ &= \mathsf{V}[\mathsf{E}(Y \mid X)] + \mathsf{E}[\mathsf{V}(Y \mid X)] . \end{aligned}$$

3. Linear regression

Consider again the setup of Section 1. We now study the problem of finding a function of the form

$$\begin{aligned} L(X) &= b_0 + b_1 X_1 + \cdots + b_k X_k \\ &= \sum_{i=0}^{k} b_i X_i = b'x \end{aligned}$$

where

$$X_0 = 1, \ b = (b_0, b_1, \ldots, b_k)' \tag{3.1}$$

$$x = (X_0, X_1, \ldots, X_k)', \tag{3.2}$$

such that the mean square prediction error

$$\mathsf{E}\left\{[Y - L(X)]^2\right\} = \mathsf{E}\left[\left(Y - b'x\right)^2\right]$$

is minimal. In other words, we wish to minimize (with respect to b) the function

$$\begin{aligned} S(b) &= \mathsf{E}\left\{\left[Y - b'x\right]^2\right\} \\ &= \mathsf{E}\left(Y^2\right) - 2b'\mathsf{E}\left(xY\right) + b'\mathsf{E}\left(xx'\right)b\,. \end{aligned}$$

It is easy to see that the optimal value of b must satisfy the equation

$$\mathsf{E}\left[x\left(Y - b'x\right)\right] = 0$$

or

$$\mathsf{E}\left(xx'\right)b = \mathsf{E}\left(xY\right)\,.$$

If we write

$$b = \begin{pmatrix} \beta_0 \\ \gamma \end{pmatrix},\ \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix},\ X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix},$$

we see that

$$\begin{bmatrix} 1 & \mathsf{E}(X)' \\ \mathsf{E}(X) & \mathsf{E}(XX') \end{bmatrix} \begin{bmatrix} \beta_0 \\ \gamma \end{bmatrix} = \begin{bmatrix} \mathsf{E}(Y) \\ \mathsf{E}(XY) \end{bmatrix},$$

hence

$$\beta_0 + \mathsf{E}(X)'\gamma = \mathsf{E}(Y) \tag{3.3}$$

$$\mathsf{E}(Y)\beta_0 + \mathsf{E}\left(XX'\right)\gamma = \mathsf{E}(XY) \tag{3.4}$$

and

$$\beta_0 = \mathsf{E}(Y) - \mathsf{E}(X)'\gamma\,.$$

Further, by the basic properties of the expectation operator,

$$\begin{aligned} \mathsf{E}\left(XX'\right) &= \mathsf{V}(X) + \mathsf{E}(X)\,\mathsf{E}(X)'\,, \\ \mathsf{E}(XY) &= \mathsf{C}(X,Y) + \mathsf{E}(X)\,\mathsf{E}(Y) \end{aligned}$$

where

$$\mathsf{V}(X) = \mathsf{E}\left\{\mathsf{E}\left[X - \mathsf{E}(X)\right]\left[X - \mathsf{E}(X)\right]'\right\}\,, \tag{3.5}$$

$$\mathsf{C}(X,Y) = \mathsf{E}\left\{\left[X - \mathsf{E}(X)\right]\left[Y - \mathsf{E}(Y)\right]'\right\}\,. \tag{3.6}$$

By the equations (3.3)-(3.6), we then see easily that

$$\begin{aligned} \mathsf{E}(X)\beta_0 + \mathsf{E}(X)\mathsf{E}(X)'\gamma &= \mathsf{E}(X)\mathsf{E}(Y) \ , \\ \mathsf{E}(X)\beta_0 + \mathsf{V}(X)\gamma + \mathsf{E}(X)\mathsf{E}(X)'\gamma &= \mathsf{C}(X,Y) + \mathsf{E}(X)\mathsf{E}(Y) \end{aligned}$$

hence

$$\mathsf{V}(X)\gamma = \mathsf{C}(X,Y) \ .$$

Thus,

$$\beta_0 = \mathsf{E}(Y) - \mathsf{E}(X)'\gamma \,, \tag{3.7}$$

$$\mathsf{V}(X)\gamma = \mathsf{C}(X,Y) \ . \tag{3.8}$$

The function

$$L(X) = \beta_0 + X'\gamma$$

is called the

linear regression of X on Y

or the

affine projection of Y on X. (3.9)

We write

$$L(X) = P_L(Y \mid X) = \beta_0 + X'\gamma$$

where β_0 and γ are any solution of the normal equations:

$$\begin{aligned} \mathsf{V}(X)\gamma &= \mathsf{C}(X,Y) \ , \\ \beta_0 &= \mathsf{E}(Y) - \mathsf{E}(X)'\gamma \,. \end{aligned}$$

If we denote by

$$\varepsilon = Y - P_L(Y \mid X)$$

the prediction error, we see easily that:

$$\begin{aligned} \mathsf{E}(\varepsilon) &= 0 \,, \\ \mathsf{C}(X,\varepsilon) &= 0 \,. \end{aligned}$$

In the language of Hilbert space theory, we can also write

$$L(X) = P_M Y = P_L(Y \mid X)$$

where

$$M = \overline{sp}\{1, X\} = \overline{sp}\{1, X_1, \ldots, X_k\} \ .$$

If

$$\det[\mathsf{V}(X)] \neq 0 \,,$$

the optimal coefficients β_0 and γ are uniquely defined :

$$\gamma = \mathsf{V}(X)^{-1}\,\mathsf{C}(X,Y)\,,\ \beta_0 = \mathsf{E}(Y) - \mathsf{E}(X)'\,\gamma\,.$$

4. Bibliographic notes

On the properties of conditional expectations, see Gouriéroux and Monfort (1995, Appendix B) and Williams (1991).

References

GOURIÉROUX, C., AND A. MONFORT (1995): *Statistics and Econometric Models, Volumes One and Two*. Cambridge University Press, Cambridge, U.K., Translated by Quang Vuong.

WILLIAMS, D. (1991): *Probability with Martingales*. Cambridge University Press, Cambridge, U.K.