Hilbert spaces *

Jean-Marie Dufour †

First version: May 1999
Revised: January 2002
This version: February 18, 2003
Compiled: February 18, 2003, 12:22pm

*This work was supported by the Canada Research Chair Program (Chair in Econometrics, Université de Montréal), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Canada Council for the Arts (Killam Fellowship), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds FCAR (Government of Québec).

† Canada Research Chair Holder (Econometrics). Centre interuniversitaire de recherche en analyse des organisations (CIRANO), Centre interuniversitaire de recherche en économie quantitative (CIREQ), and Département de sciences économiques, Université de Montréal. Mailing address: Département de sciences économiques, Université de Montréal, C.P. 6128 succursale Centre-ville, Montréal, Québec, Canada H3C 3J7. TEL: 1 514 343 2400; FAX: 1 514 343 5831; e-mail: jean.marie.dufour@umontreal.ca. Web page: http://www.fas.umontreal.ca/SCECO/Dufour .

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List of Definitions, Propositions and Theorems

1. Inner-product spaces

1.1 Definition INNER PRODUCT SPACE. *Let $\mathcal{H}$ be a vector space on a set scalars S, where $S = \mathbb{R}$ or $S = \mathbb{C}$. An* inner product *on H is an application which associates to each pair of elements x and y in $\mathcal{H}$ a complex number $\langle x, y \rangle$ such that, for all x, y, $z \in \mathcal{H}$,*

(a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;

(c) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, *for all* $\alpha \in S$;

(d) $\langle x, x \rangle \geq 0$;

(e) $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{H}$, the pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called an inner product space*, and the elements of $\mathcal{H}$ are also called elements of the inner-product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. When there is no ambiguity on the definition of the inner product, the inner-product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ may simply be denoted $\mathcal{H}$.*

1.2 Remark If $S = \mathbb{R}$, condition (a) reduces to

(a)$'$ $\langle x, y \rangle = \langle y, x \rangle$.

1.3 Definition NORM ASSOCIATED WITH AN INNER PRODUCT. *The* norm *of an element x of an inner-product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is defined by*

$$\|x\| = \sqrt{\langle x, x \rangle} \, . \tag{1.1}$$

1.4 Example EUCLIDEAN SPACE. $\mathbb{R}^n$ with the usual scalar product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \tag{1.2}$$

where $x = (x_1, \ldots, x_x)'$ $y = (y_1, \ldots, y_x)'$, where $x_i \in \mathbb{R}$ and $y_j \in \mathbb{R}$ for all i and j, is an inner-product space whose norm is the usual Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \, . \tag{1.3}$$

1.5 Example COMPLEX EUCLIDEAN SPACE. $\mathbb{C}^n$ with the scalar product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i \tag{1.4}$$

where $x = (x_1, \dots, x_n)'$ $y = (y_1, \dots, y_n)'$, where $x_i \in \mathbb{C}$ and $y_j \in \mathbb{C}$ for all i and j, is an inner-product space. The associated norm is:

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} . \tag{1.5}$$

1.6 Example REAL-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2 = L^2(\Omega, \mathcal{A}, P)$ of all the random variables $X : \Omega \to \mathbb{R}$ such that $E(X^2) < \infty$ with, for any $X, Y \in L^2$,

$$\langle X, Y\rangle = E(XY) \tag{1.6}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(X^2)]^{1/2} . \tag{1.7}$$

1.7 Example COMPLEX-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$ of all the random variables $X : \Omega \to \mathbb{C}$ such that $E(|X|^2) < \infty$ with

$$\langle X, Y\rangle = E(X\overline{Y}) \tag{1.8}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(|X|^2)]^{1/2} . \tag{1.9}$$

1.8 Proposition CAUCHY-SCHWARZ INEQUALITY. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner-product space. Then, for all $x, y \in \mathcal{H}$,*

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \tag{1.10}$$

and

$$|\langle x, y\rangle| = \|x\|\,\|y\| \;\Leftrightarrow\; x = \left(\frac{\langle x, y\rangle}{\langle y, y\rangle}\right) y . \tag{1.11}$$

1.9 Definition ANGLE AND ORTHOGONALITY. *Let x and y be two elements of an inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The* angle *between the vectors x and y is defined by*

$$\theta = \cos^{-1}[\langle x, y\rangle / (\|x\|\,\|y\|)] . \tag{1.12}$$

x and y are said to be orthogonal *(denoted $x \perp y$) if and only if*

$$\langle x, y\rangle = 0 . \tag{1.13}$$

1.10 Proposition PROPERTIES OF THE NORM. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then the following properties hold for all $x, y \in \mathcal{H}$:*

(*a*) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$;

(b) $\|x+y\| \leq \|x\| + \|y\|$; (*Triangle inequality*)

(c) $\|\alpha x\| \leq |\alpha| \, \|x\|$, *for all* $\alpha \in \mathbb{C}$;

(d) $\|x\| \geq 0$;

(e) $\|x\| = 0$ *if and only if* $x = 0$;

(f) $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$. (*Parallelogram law*)

1.11 Proposition CONVERGENCE IN NORM. *Let* $\{x_n : n = 1,\, 2,\, ...\}$ *be a sequence of elements of an iner product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. *We say that* x_n *converges to* $x \in \mathcal{H}$ *if and only if*

$$\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 . \tag{1.14}$$

1.12 Proposition CONTINUITY OF INNER PRODUCT. *Let* $\{x_n : n = 1,\, 2,\, ...\}$ *and* $\{y_n : n = 1,\, 2,\, ...\}$ *be two sequences of elements of an inner-product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *such that* $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$ *and* $\|y_n - y\| \underset{n\to\infty}{\longrightarrow} 0$, *where* $x,\, y \in \mathcal{H}$ *.Then*

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\| \tag{1.15}$$

and

$$\langle x_n, y_n \rangle \underset{n\to\infty}{\longrightarrow} \langle x, y \rangle \ . \tag{1.16}$$

1.13 Example MEAN SQUARE CONVERGENCE. Let $\{X_n : n = 1,\, 2,\, ...\}$ a sequence of real random variables in $L^2 \equiv L^2(\Omega,\, \mathcal{A},\, P)$. If we define $\langle X, Y \rangle = E(XY)$, then the convergence in norm represents mean-square convrgence:

$$\|X_n - X\| \underset{n\to\infty}{\longrightarrow} 0 \ \Leftrightarrow \ E[(X_n - X)^2] \underset{n\to\infty}{\longrightarrow} 0 . \tag{1.17}$$

2. Hilbert spaces

2.1 Definition CAUCHY SEQUENCE. *Let* $\{x_n : n = 1,\, 2,\, ...\}$ *be a sequence of elements of an inner-product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. *Then* x_n *is a* Cauchy sequence *of* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *if and only if*

$$\|x_n - x_m\| \underset{m,\,n\to\infty}{\longrightarrow} 0 . \tag{2.1}$$

2.2 Definition HILBERT SPACE. *Let* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *an inner-product space. Then* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$*is a* Hilbert space *if and only if every Cauchy sequence of* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *converges in norm to an element of* $\mathcal{H}$,

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 \ \Rightarrow \ \exists x \in \mathcal{H} \text{ such that } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 . \tag{2.2}$$

2.3 Proposition NORM CONVERGENCE AND CAUCHY CRITERION. *Let $\{x_n : n = 1, 2, ...\}$ be a sequence of elements of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then x_n converges in norm if and only if x_n is a Cauchy sequence.*

2.4 Theorem COMPLETENESS OF $L^2(\Omega, \mathcal{A}, P)$. *The inner-product space $L^2(\Omega, \mathcal{A}, P)$ with $\langle X, Y\rangle = E(XY)$ is complete and thus constitutes a Hilbert space.*

PROOF. See Brockwell and Davis (1991, Section 2.10). ∎

3. Projection theorems

3.1 Definition CLOSED SUBSPACE. *A linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is a* closed subspace *of $\mathcal{H}$ if and only if $\mathcal{M}$ contains all its limit points, i.e.*

$$x_n \in \mathcal{M}\,,\ \forall n\,,\ \text{and}\ \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \ \Rightarrow\ x \in \mathcal{M}\,. \tag{3.1}$$

3.2 Definition ORTHOGONAL COMPLEMENT. *The orthogonal complement $\mathcal{M}^\perp$ of a subset $\mathcal{M}$ of $\mathcal{H}$ is the set of all $x \in \mathcal{H}$ which are orthogonal to all the elements of $\mathcal{M}$, i.e.*

$$x \in \mathcal{M}^\perp \Leftrightarrow \langle x, y\rangle = 0\ ,\ \forall y \in \mathcal{M}\,. \tag{3.2}$$

3.3 Proposition CLOSURE OF ORTHOGONAL COMPLEMENT. *If $\mathcal{M}$ is any subset of a Hilbert space $\mathcal{H}$, then $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$.*

3.4 Theorem PROJECTION THEOREM. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then, for any $x \in \mathcal{H}$,*

(*a*) *there is a unique element $\widehat{x} \in \mathcal{M}$ such that*

$$\|x - \widehat{x}\| = \inf_{y\in\mathcal{M}} \|x - y\| \tag{3.3}$$

and

(*b*)

$$\begin{aligned} &\widehat{x} \in \mathcal{M} \text{ and } \|x - \widehat{x}\| = \inf_{y\in\mathcal{M}} \|x - y\| \\ &\Leftrightarrow \widehat{x} \in \mathcal{M} \text{ and } x - \widehat{x} \in \mathcal{M}^\perp\,. \end{aligned} \tag{3.4}$$

3.5 Corollary EXISTENCE OF PROJECTION MAPPING OF $\mathcal{H}$ ONTO $\mathcal{M}$. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$, and let I be the identity mapping on $\mathcal{H}$. Then there is a unique mapping $P_{\mathcal{M}}$ of $\mathcal{H}$ onto $\mathcal{M}$ such that $I - P_{\mathcal{M}}$ maps $\mathcal{H}$ onto $\mathcal{M}^\perp$. $P_{\mathcal{M}}$ is called the projection mapping of $\mathcal{H}$ onto $\mathcal{M}$.*

3.6 Theorem PROPERTIES OF PROJECTION MAPPINGS. *Let $\mathcal{H}$ be a Hilbert space and $P_{\mathcal{M}}$ the projection mapping onto the closed subspace $\mathcal{M}$. Then the following properties hold for all $x, y \in \mathcal{H}$:*

(*a*) *each element $x \in \mathcal{H}$ has a unique representation as the sum of an element of $\mathcal{M}$ and an element of $\mathcal{M}^{\perp}$,*

$$x = P_{\mathcal{M}}x + (I - P_{\mathcal{M}})x \; ; \tag{3.5}$$

(*b*) $\|x\|^2 = \|P_{\mathcal{M}}x\|^2 + \|(I - P_{\mathcal{M}})x\|^2$;

(*c*) $P_{\mathcal{M}}(\alpha x + \beta y) = \alpha P_{\mathcal{M}}x + \beta P_{\mathcal{M}}y$, *for all* α , $\beta \in S$;

(*d*) $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow P_{\mathcal{M}}x_n \underset{n\to\infty}{\longrightarrow} P_{\mathcal{M}}x$;

(*e*) $x \in \mathcal{M} \Leftrightarrow P_{\mathcal{M}}x = x$;

(*f*) $x \in \mathcal{M}^{\perp} \Leftrightarrow P_{\mathcal{M}}x = 0$;

(*g*) *if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two subspaces of $\mathcal{H}$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then*

$$P_{\mathcal{M}_1}(P_{\mathcal{M}_2}x) = P_{\mathcal{M}_1}x \; . \tag{3.6}$$

4. Orthonormal sets

4.1 Definition CLOSED SPAN. *Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. The closed span of $\mathcal{M}$, denoted $\overline{sp}(\mathcal{M})$, is the smallest closed subspace of $\mathcal{H}$ that contains $\mathcal{M}$. If $\mathcal{M} = \{e_t : t \in T\}$, we can also denote $\overline{sp}(\mathcal{M})$ by $\overline{sp}\{e_t : t \in T\}$.*

4.2 Proposition CLOSED SPAN OF A FINITE SET. *The closed span of a finite set $\mathcal{M} = \{x_1, ..., x_n\}$ is the set of all linear combinations*

$$y = \alpha_1 x_1 + ... + \alpha_n x_n \; , \; \alpha_1 \, , \, ... \, , \, \alpha_n \in S \tag{4.1}$$

where $S = \mathbb{R}$ or $S = \mathbb{C}$.

4.3 Definition ORTHOGONAL AND ORTHONORMAL SETS. *Let $\{e_t : t \in T\}$ be a subset of an inner-product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then $\{e_t : t \in T\}$ is said to be* orthogonal *if and only if*

$$\langle e_s, e_t\rangle = 0 \, , \text{ for } s \neq t \; . \tag{4.2}$$

Further, $\{e_t : t \in T\}$ is said to be orthonormal *if and only it is orthogonal and $\langle e_t, e_t\rangle = 1$ for all $t \in T$.*

4.4 Theorem PROJECTIONS ON FINITE ORTHONORMAL SETS. *If $\{e_1, \dots, e_n\}$ is an orthonormal subset of a Hilbert space $\mathcal{H}$ and $\mathcal{M} = \overline{sp}\{e_1, \dots, e_n\}$, then the following properties hold for all $x \in \mathcal{H}$:*

(a) $P_{\mathcal{M}}x = \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i$;

(b) $\|P_{\mathcal{M}}x\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$;

(c) $\left\|x - \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i\right\| \le \left\|x - \sum_{i=1}^{n} c_i e_i\right\|$, *for all* $c_1, \dots, c_n \in S$;

(d)
$$\left\|x - \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i\right\| = \left\|x - \sum_{i=1}^{n} c_i e_i\right\| \Leftrightarrow c_i = \langle x, e_i\rangle\ ,\ i = 1, \dots, n\,; \tag{4.3}$$

(e) $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$. (*Bessel's inequality*)

4.5 Remark The coefficients $c_i = \langle x, e_i\rangle$, $i = 1, \dots, n$, are called the *Fourier coefficients* relative to the set $\{e_1, \dots, e_n\}$.

4.6 Corollary PROJECTION ON A UNIDIMENSIONAL SUBSPACE. *Suppose $\mathcal{M} = \overline{sp}\{e_1\}$ where $e_1 \in \mathcal{H}$ and $e_1 \neq 0$. Then*

$$P_{\mathcal{M}}x = \frac{\langle x, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = \frac{\langle x, e_1\rangle}{\|e_1\|^2} e_1\ . \tag{4.4}$$

4.7 Corollary PROJECTIONS ON FINITE ORTHOGONAL SETS. *Let $\{e_1, \dots, e_n\}$ be a finite set of non-zero orthogonal vectors in a Hilbert space $\mathcal{H}$, let $\mathcal{M} = \overline{sp}\{e_1, \dots, e_n\}$, and let $\beta_i(x) = \langle x, e_i\rangle / \langle e_i, e_i\rangle$, $i = 1, \dots, n$. Then the following properties hold for all $x \in \mathcal{H}$:*

(a) $P_{\mathcal{M}}x = \sum_{i=1}^{n} \beta_i(x) e_i$;

(b) $\|P_{\mathcal{M}}x\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 / \langle e_i, e_i\rangle = \sum_{i=1}^{n} |\beta_i(x)|^2\, \|e_i\|^2$;

(c) $\left\|x - \sum_{i=1}^{n} \beta_i(x) e_i\right\| \le \left\|x - \sum_{i=1}^{n} c_i e_i\right\|$, *for all* $c_1, \dots, c_n \in S$;

(d)
$$\left\|x - \sum_{i=1}^{n} \beta_i(x) e_i\right\| = \left\|x - \sum_{i=1}^{n} c_i e_i\right\| \Leftrightarrow c_i = \beta_i\ ,\ i = 1, \dots, n\,; \tag{4.5}$$

(e) $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\beta_i(x)|^2 \|e_i\|^2 \le \|x\|^2$. (*Bessel's inequality*)

4.8 Definition SEPARABILITY. *The Hilbert space $\mathcal{H}$ is* separable *if and only if $\mathcal{H} = \overline{sp}\{e_t : t \in T\}$ where T is a finite or countable infinite set.*

4.9 Theorem ORTHONORMAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let $\mathcal{H} = \overline{sp}\{e_1, e_2, ...\}$ be a separable Hilbert space where $\{e_1, e_2, ...\}$ is an orthonormal set. Then the following properties hold for all $x \in \mathcal{H}$:*

(a) *the set of all finite linear combinations of $\{e_1, e_2, ...\}$ is dense in $\mathcal{H}$, i.e., for any $\varepsilon > 0$, there is an $n > 0$ and constants $c_1, ..., c_n \in S$ such that*

$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon \; ; \tag{4.6}$$

(b) $x = \sum_{i=1}^{\infty} \langle x, e_i\rangle \, e_i$, i.e.

$$\left\| x - \sum_{i=1}^{n} \langle x, e_i\rangle \, e_i \right\| \underset{n\to\infty}{\longrightarrow} 0 \; ; \tag{4.7}$$

(c) $\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$;

(d) $\langle x, y\rangle = \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle$, $\forall\, y \in \mathcal{H}$; (*Parseval's identity*)

(e) $x = 0 \Leftrightarrow \langle x, e_i\rangle = 0$, , $i = 1, 2, ...$.

4.10 Corollary ORTHOGONAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let $\mathcal{H} = \overline{sp}\{e_1, e_2, ...\}$ be a separable Hilbert space where $\{e_1, e_2, ...\}$ is a set of non-zero orthogonal vectors, and let $\beta_i(x) = \langle x, e_i\rangle / \langle e_i, e_i\rangle$, $i = 1, ..., n$. Then the following properties hold for all $x \in \mathcal{H}$:*

(a) *the set of all finite linear combinations of $\{e_1, e_2, ...\}$ is dense in $\mathcal{H}$, i.e., for any $\varepsilon > 0$, there is an $n > 0$ and constants $c_1, ..., c_n \in S$ such that*

$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon \; ; \tag{4.8}$$

(*b*) $x = \sum\limits_{i=1}^{\infty} \beta_i(x) e_i$, *i.e.*

$$\left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| \underset{n\to\infty}{\longrightarrow} 0 \; ; \tag{4.9}$$

(*c*) $\|x\|^2 = \sum\limits_{i=1}^{n} |\langle x, e_i\rangle|^2 / \|e_i\|^2 = \sum\limits_{i=1}^{n} |\langle x, e_i\rangle|^2 / \langle e_i, e_i\rangle = \sum\limits_{i=1}^{n} |\beta_i(x)|^2 \, \|e_i\|^2$;

(*d*)
$$\begin{aligned} \langle x, y\rangle &= \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle / \|e_i\|^2 = \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle / \langle e_i, e_i\rangle \\ &= \sum_{i=1}^{\infty} \beta_i(x)\overline{\beta_i(y)} \, \|e_i\|^2 \; , \; \forall\, y \in \mathcal{H} \; ; \end{aligned}$$

(*Parseval's identity*)

(*e*) $x = 0 \;\Leftrightarrow\; \langle x, e_i\rangle = 0 \,,\; i = 1, 2, \ldots$.

4.11 Corollary PROJECTIONS IN SEPARABLE HILBERT SPACES. *Under the assumptions of Theorem* **4.9**, *we have for each* $x \in \mathcal{H}$:

(*a*) *for any* $\varepsilon > 0$, *there is an* $n > 0$ *such that*

$$\|x - P_{E_n} x\| < \varepsilon \tag{4.10}$$

where $E_n = \overline{sp}\{e_1, \ldots, e_n\}$;

(*b*) $P_{E_n} x \underset{n\to\infty}{\longrightarrow} x$.

5. Conditional expectations

5.1 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A CLOSED SUBSPACE. *Let* $\mathcal{M}$ *be a closed subspace of* $L^2 \equiv L^2(\Omega, \mathcal{A}, P)$ *containing the constant functions and let* $Y \in L^2$. *Then the conditional expectation of* Y *given* $\mathcal{M}$ *is the projection of* Y *on* $\mathcal{M}$:

$$E_{\mathcal{M}} Y = P_{\mathcal{M}} Y \; . \tag{5.1}$$

5.2 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A RANDOM VECTOR. *Let* $X = (X_1, \ldots, X_k)'$ *be a vector of random variables in* $L^2 \equiv L^2(\Omega, \mathcal{A}, P)$ *and let* $L^2(X)$ *be the closed subspace of* $L^2(\Omega, \mathcal{A}, P)$ *consisting of all the random variables in* L^2 *of the form* $\phi(X)$ *for some function* ϕ. *Then the conditional expectation of* Y *given* X *is the projection of* Y *on* $L^2(X)$:

$$E(Y|X) = E(Y|X_1, \ldots, X_k) = P_{L^2(X)} Y \; . \tag{5.2}$$

6. Sources and additional references

A good summary of Hilbert space theory aimed at applications in time series analysis may be found in Brockwell and Davis (1991, Chapter 2). Other good reviews appear in: Debnath and Mikusiński (1990) for general applications, Small and McLeish (1994) for applications in statistical theory, and Young (1988) for a more mathematically oriented presentation.

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