Introduction to stochastic processes *

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Contents

1. Basic notions

1.1. Probability space

1.1.1 Definition *A* probability space *is a triplet* $(\Omega, \mathcal{A}, P)$ *where*

(1) Ω *is the set of all possible results of an experiment;*

(2) $\mathcal{A}$ *is class of subsets of* Ω *(called* events*) forming a* $\sigma-$*algebra,* $i.e.$

(i) $\Omega \in \mathcal{A}$,

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,

(iii) $\overset{\infty}{\underset{j=1}{\cup}} A_j \in \mathcal{A}$, *for any sequence* $\{A_1,\, A_2, ...\} \subseteq \mathcal{A}$ *;*

(3) $P : \mathcal{A} \to [0,\ 1]$ *is a function which assigns to each event* $A \in \mathcal{A}$ *a number* $P(A) \in [0,\ 1]$, *called the probability of* A *and such that*

(i) $P(\Omega) = 1$,

(ii) *if* $\{A_j\}_{j=1}^{\infty}$ *is a sequence of disjoint events, then* $P(\overset{\infty}{\underset{j=1}{\cup}} A_j) = \sum\limits_{j=1}^{\infty} P(A_j)$.

1.2. Real random variable

1.2.1 Definition (heuristic) *A* real random variable X *is a variable with real values whose behavior can be described by a probability distribution. Usually, this probability distribution is described by a distribution function:*

$$F_X(x) = P[X \leq x]\ . \tag{1.1}$$

1.2.2 Definition (formal) *A* real random variable X *is a function* $X : \Omega \to \mathbb{R}$ *such that*

$$X^{-1}((-\infty,\ x]) \equiv \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{A}, \forall x \in \mathbb{R}, \quad \textit{(measurable function)}.$$

The probability law of X *is defined by*

$$F_X(x) = P[X^{-1}((-\infty, x])]\ . \tag{1.2}$$

1.3. Stochastic process

1.3.1 Definition *Let T be a non-empty set. A stochastic process on T is a collection of $r.v.'s$ $X_t : \Omega \to \mathbb{R}$ such that to each element $t \in T$ is associated a $r.v.$ X_t. The process can be written $\{X_t : t \in T\}$. If $T = \mathbb{R}$ (real numbers), we have a process in continuous time. If $T = \mathbb{Z}$ (integers) or $T \subseteq \mathbb{Z}$, we have discrete time process.*

The set T can be finite or infinite, but usually it is assumed to be infinite. In the sequel, we shall be mainly interested by processes for which T is a right-infinite interval of integers: $i.e.$, $T = (n_0, \infty)$ where $n_0 \in \mathbb{Z}$ or $n_0 = -\infty$. We can also consider $r.v.'s$ which take their values in more general spaces, $i.e.$

$$X_t : \Omega \to \Omega_0$$

where Ω_0 is any non-empty set. Unless stated otherwise, we shall limit ourselves to the case where $\Omega_0 = \mathbb{R}$.

To observe a time series is equivalent to observing a realization of a process $\{X_t : t \in T\}$ or a portion of such a realization: given $(\Omega, \mathcal{A}, P)$, $\omega \in \Omega$ is first drawn and then the variables $X_t(\omega)$, $t \in T$, are associated with it. Each realization is determined in one shot by ω.

The probability law of a stochastic process $\{X_t : t \in T\}$ where $T \subseteq \mathbb{R}$ can be described by specifying, for each subset $\{t_1, t_2, \dots, t_n\} \subseteq T$ (where $n \geq 1$), the joint distribution function of $(X_{t_1}, \dots, X_{t_n})$:

$$F(x_1, \dots, x_n; t_1, \dots, t_n) = P[X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n] . \quad (1.1)$$

This follows from Kolmogorov's theorem [see Brockwell and Davis (1991, Chapter 1)].

1.4. L_r spaces

1.4.1 Definition *Let r be a real number. L_r is the set of real random variables X defined on $(\Omega, \mathcal{A}, P)$ such that $E[|X|^r] < \infty$.*

The space L_r is always defined with respect to a probability space $(\Omega, \mathcal{A}, P)$. L_2 is the set of $r.v.'s$ on $(\Omega, \mathcal{A}, P)$ whose second moments are finite (*square-integrable variables*). A stochastic process $\{X_t : t \in T\}$ is in L_r iff $X_t \in L_r$, $\forall t \in T$, $i.e.$

$$E[|X_t|^r] < \infty \, , \forall t \in T \, . \quad (1.1)$$

The properties of moments of $r.v.'s$ are summarized in Dufour (1999b).

2. Stationary processes

In general, the variables of a process $\{X_t : t \in T\}$ are not identically distributed nor independent. In particular, if we suppose that $E(X_t^2) < \infty$, we have

$$E(X_t) = \mu_t \; , \tag{2.1}$$

$$Cov(X_{t_1}, X_{t_2}) = E[(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})] = C(t_1, t_2) \; . \tag{2.2}$$

The means, variances and covariances of the variables of the process depend on their position in the series. The behavior of X_t can change with time. The function $C : T \times T \to \mathbb{R}$ is called the *covariance function* of the process $\{X_t : t \in T\}$.

In this section, we will study the case where T is an right-infinite interval of integers.

2.1 Assumption *(Process on an interval of integers).*

$$T = \{t \in \mathbb{Z} : t > n_0\} \; , \quad \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}. \tag{2.3}$$

2.2 Definition *(Strictly stationary process) : A stochastic process $\{X_t : t \in T\}$ is strictly stationary (SS) iff the joint probability law of the vector $(X_{t_1+k},\ X_{t_2+k},\ \ldots\ ,\ X_{t_n+k})'$ is identical with the one of $(X_{t_1},\ X_{t_2},\ \ldots\ ,\ X_{t_n})'$, for any finite subset $\{t_1, t_2, \ldots , t_n\} \subseteq T$ and for any integer $k \geq 0$. To indicate that $\{X_t : t \in T\}$ is SS, we will write $\{X_t : t \in T\} \sim SS$ or $X_t \sim SS$.*

2.3 Proposition *If the process $\{X_t : t \in T\}$ is SS, then the joint probability law of the vector $(X_{t_1+k},\ X_{t_2+k},\ \ldots\ ,\ X_{t_n+k})'$ is identical to the one of $(X_{t_1},\ X_{t_2},\ \ldots\ ,\ X_{t_n})'$, for any finite subset $\{t_1, t_2, \ldots , t_n\}$ and any integer $k > n_0 - \min\{t_1, \ldots , t_n\}$.*

2.4 Proposition *(Strict stationarity of a process on the integers). A process $\{X_t : t \in \mathbb{Z}\}$ is SS iff the joint probability law of $(X_{t_1+k}, X_{t_2+k}, \ldots , X_{t_n+k})'$ is identical with the law of $(X_{t_1}, X_{t_2}, \ldots , X_{t_n})'$, for any subset $\{t_1, t_2, \ldots , t_n\} \subseteq \mathbb{Z}$ and any integer k.*

Suppose $E(X_t^2) < \infty$, for any $t \in T$. If the process $\{X_t : t \in T\}$ is SS, we see easily that

$$E(X_s) = E(X_t) \;, \forall s, t \in T \;, \tag{2.4}$$

$$E(X_s X_t) = E(X_{s+k} X_{t+k}) \;, \forall s, t \in T, \forall k \geq 0 \;. \tag{2.5}$$

Furthermore, since

$$Cov(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t) \;, \tag{2.6}$$

we also have

$$Cov(X_s, X_t) = Cov(X_{s+k}, X_{t+k}) \;, \forall s, t \in T \;, \forall k \geq 0 \;. \tag{2.7}$$

The conditions (2.4) and (2.5) are equivalent to the conditions (2.4) and (2.7). The mean of X_t is constant and the covariance between any two variables of the process only depends on the distance between the variables, but not their position in the series.

2.5 Definition *(Second-order stationary process). A stochastic process* $\{X_t : t \in T\}$ *is* second-order stationary *(S2) iff*

(1) $E(X_t^2) < \infty, \forall t \in T,$
(2) $E(X_s) = E(X_t), \forall s, t \in T,$
(3) $Cov(X_s, X_t) = Cov(X_{s+k}, X_{t+k}), \forall s, t \in T, \forall k \geq 0 \;.$

If $\{X_t : t \in T\}$ *is S2, we write* $\{X_t : t \in T\} \sim S2$ *or* $X_t \sim S2$.

2.6 Remark Instead of *second-order stationary*, one also says *weakly stationary* (WS).

2.7 Proposition *(Relation between strict stationarity and second-order stationarity). If the process* $\{X_t : t \in T\}$ *is strictly stationary and* $E(X_t^2) < \infty$ *for any* $t \in T$*, then the process* $\{X_t : t \in T\}$ *is second-order stationary.*

2.8 Proposition *(Existence of an autocovariance function). If the process* $\{X_t : t \in T\}$ *is second-order stationary, then there exists a function* $\gamma : \mathbb{Z} \to \mathbb{R}$ *such that*

$$Cov(X_s, X_t) = \gamma(t - s) \;, \forall s, t \in T. \tag{2.8}$$

The function γ is called the autocovariance function of the process $\{X_t : t \in T\}$ and $\gamma(k)$, for k given, the lag-k autocovariance of the process $\{X_t : t \in T\}$.

PROOF: Let $r \in T$ any element of T. Since the process $\{X_t : t \in T\}$ is S2, we have, for any $s, t \in T$ such that $s \leq t$,

$$Cov(X_r, X_{r+t-s}) = Cov(X_{r+s-r}, X_{r+t-s+s-r}) = Cov(X_s, X_t)\ ,\ \text{if } s \geq r, \tag{2.9}$$

$$Cov(X_s, X_t) = Cov(X_{s+r-s}, X_{t+r-s}) = Cov(X_r, X_{r+t-s})\ ,\ \text{if } s < r. \tag{2.10}$$

Further, in the case where $s > t$, we have

$$Cov(X_s, X_t) = Cov(X_t, X_s) = Cov(X_r, X_{r+s-t})\ . \tag{2.11}$$

Thus

$$Cov(X_s, X_t) = Cov(X_r, X_{r+|t-s|}) = \gamma(t-s)\ . \tag{2.12}$$

$Q.E.D.$

2.9 Proposition *(Properties of the autocovariance function). Let $\{X_t : t \in T\}$ be a second-order stationary process. The autocovariance function $\gamma(k)$ of the process $\{X_t : t \in T\}$ satisfies the following properties:*

(1) $\gamma(0) = Var(X_t) \geq 0\ ,\ \forall t \in T$;

(2) $\gamma(k) = \gamma(-k)\ ,\ \forall k \in \mathbb{Z}$ ($i.e.$, $\gamma(k)$ *is an even function of* k);

(3) $|\gamma(k)| \leq \gamma(0)\ ,\ \forall k \in \mathbb{Z}$;

(4) *the function* $\gamma(k)$ *is positive semi-definite,* $i.e.$ $\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \gamma(t_i - t_j) \geq 0$, *for any positive integer* N *and for all the vectors* $a = (a_1, \ldots, a_N)' \in \mathbb{R}^N$ *and* $\tau = (t_1, \ldots, t_N)' \in T^N$;

(5) *any* $N \times N$ *matrix of the form*

$$\varGamma_N = [\gamma(j-i)]_{i,\, j=1,\, \ldots\, ,\, N} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{N-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{N-1} & \gamma_{N-2} & \gamma_{N-3} & \cdots & \gamma_0 \end{bmatrix} \tag{2.13}$$

is positive semi-definite, where $\gamma_k \equiv \gamma(k)$.

2.10 Proposition *(Existence of an autocorrelation function). If the process* $\{X_t : t \in T\}$ *is second-order stationary, then there exists a function* $\rho : \mathbb{Z} \to [-1,\ 1]$ *such that*

$$\rho(t-s) = Corr(X_s, X_t) = \gamma(t-s)/\gamma(0)\ , \forall s, t \in T\,, \tag{2.14}$$

where $0/0 \equiv 1$. *The function* ρ *is called the* autocorrelation function *of the process* $\{X_t : t \in T\}$, *and* $\rho(k)$, *for* k *given, the* lag-k autocorrelation *of the process* $\{X_t : t \in T\}$.

2.11 Proposition *(Properties of the autocorrelation function). Let* $\{X_t : t \in T\}$ *be a second-order stationary process. The autocorrelation function* $\rho(k)$ *of the process* $\{X_t : t \in T\}$ *satisfies the following properties:*

(1) $\rho(0) = 1$;

(2) $\rho(k) = \rho(-k)\ , \forall k \in \mathbb{Z}\,$;

(3) $|\rho(k)| \leq 1, \forall k \in \mathbb{Z}\,$;

(4) *the function* $\rho(k)$ *is positive semi-definite,* $i.e.$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \rho(t_i - t_j) \geq 0 \tag{2.15}$$

for any positive integer N *and for all the vectors* $a = (a_1,\ \dots\ ,\ a_N)' \in \mathbb{R}^N$ *and* $\tau = (t_1,\ \dots\ ,\ t_N)' \in T^N$;

(5) any $N \times N$ *matrix of the form*

$$R_N = \frac{1}{\gamma_0}\Gamma_N = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} & \cdots & 1 \end{bmatrix} \tag{2.16}$$

is positive semi-definite, where $\gamma_0 = Var(X_t)$ *and* $\rho_k \equiv \rho(k)$.

2.12 Theorem *(Characterization of autocovariance functions) : An even function* $\gamma : \mathbb{Z} \to \mathbb{R}$ *is positive semi-definite iff* $\gamma(.)$ *is the autocovariance function of a second-order stationary process* $\{X_t : t \in \mathbb{Z}\}$.

PROOF: See Brockwell and Davis (1991, Chapter 2).

2.13 Corollary *(Characterization of autocorrelation functions). An even function* $\rho : \mathbb{Z} \to [-1,\ 1]$ *is positive semi-definite iff* ρ *is the autocorrelation function of a second-order stationary process* $\{X_t : t \in \mathbb{Z}\}$.

2.14 Definition *(Deterministic process). Let* $\{X_t : t \in T\}$ *be a stochastic process,* $T_1 \subseteq T$ *and* $I_t = \{X_s : s \leq t\}$. *We say that the process* $\{X_t : t \in T\}$ *is* deterministic *on* T_1 *iff there exists a collection of functions* $\{g_t(I_{t-1}) : t \in T_1\}$ *such that* $X_t = g_t(I_{t-1})$ *with probability 1,* $\forall t \in T_1$.

A deterministic process is a process which can be perfectly predicted form its own past (at points where it is deterministic).

2.15 Proposition *(Criterion for a deterministic process). Let* $\{X_t : t \in T\}$ *be a second-order stationary process, where* $T = \{t \in \mathbb{Z} : t > n_0\}$ *and* $n_0 \in \mathbb{Z} \cup \{-\infty\}$, *and let* $\gamma(k)$ *its autocovariance function. If there exists an integer* $N \geq 1$ *such that the matrix* Γ_N *is singular [where* Γ_N *is defined in Proposition* **2.9***], then the process* $\{X_t : t \in T\}$ *is deterministic for* $t > n_0 + N - 1$. *In particular, if* $Var(X_t) = \gamma(0) = 0$, *the process is deterministic for* $t \in T$.

For a second-order indeterministic stationary process en any $t \in T$, all the matrices $\Gamma_N, N \geq 1$, are invertible.

2.16 Definition *(Stationary of order* m*). Let* m *be a non-negative integer. A stochastic process* $\{X_t : t \in T\}$ *is stationary of order* m *iff*

(1) $E(|X_t|^m) < \infty$, $\forall t \in T$,
and

(2) $E[X_{t_1}^{m_1} X_{t_2}^{m_2} \ldots X_{t_n}^{m_n}] = E\left[X_{t_1+k}^{m_1} X_{t_2+k}^{m_2} \ldots X_{t_n+k}^{m_n}\right]$
for any $k \geq 0$, *any subset* $\{t_1, \ldots, t_n\} \in T^N$ *and all the non-negative integers* $m_1, \ldots, m_n$ *such that* $m_1 + m_2 + \ldots + m_n \leq m$.

If $m = 1$, the mean is constant, but not necessarily the other moments. If $m = 2$, the process is second-order stationary.

2.17 Definition *(Asymptotically stationary process of order m). Let m a non-negative integer. A stochastic process $\{X_t : t \in T\}$ is asymptotically stationary of order m iff*

(1) *there exists an integer N such that $(|X_t|^m) < \infty$,for $t \geq N$,*
and

(2) $\lim_{t_1 \to \infty} \left\{ E\left(X_{t_1}^{m_1} X_{t_1+\Delta_2}^{m_2} ... X_{t_1+\Delta_n}^{m_n}\right) - E\left(X_{t_1+k}^{m_1} X_{t_1+\Delta_2+k}^{m_2} ... X_{t_1+\Delta_n+k}^{m_n}\right)\right\} = 0$
for any $k \geq 0,\ t_1 \in T$, all the positive integers $\Delta_2, \Delta_3, ... , \Delta_n$ such that $\Delta_2 < \Delta_3 < ... < \Delta_n$, and all the non-negative integers $m_1,\ ... ,\ m_n$ such that $m_1 + m_2 + ... + m_n \leq m$.

3. Some important models

In this section, we will again assume that T is a right-infinite interval integers (Assumption 2.1) :

$$T = \{t \in \mathbb{Z} : t > n_0\}\ ,\ \text{where } n_0 \in \mathbb{Z} \cup \{-\infty\}\,. \tag{3.1}$$

3.1. Noise models

3.1.1 Definition Sequence of independent $r.v.'s$: *process $\{X_t : t \in T\}$ such that the variables X_t are mutually independent. We write*

$$X_t : t \in T\}\ \sim IND \text{ or } \{X_t\}\ \sim\ IND; \tag{3.2}$$

$$\{X_t : t \in T\}\ \sim\ IND(\mu_t) \text{ or } E(X_t) = \mu_t; \tag{3.3}$$

$$\{X_t : t \in T\}\ \sim\ IND(\mu_t, \sigma_t^2),\ \text{if } E(X_t) = \mu_t \text{ and } Var(X_t) = \sigma_t^2. \tag{3.4}$$

3.1.2 Definition Random sample*: sequence of independent and identically distributed $(i.i.d.)$ $r.v.'s$. We write*

$$\{X_t : t \in T\}\ \sim IID\ . \tag{3.5}$$

A random sample is a SS process. If $E(X_t^2) < \infty$, for any $t \in T$, the process is S2. In this case, we write

$$\{X_t : t \in T\} \sim\ IID(\mu, \sigma^2)\ ,\ \text{if } E(X_t) = \mu \text{ and } V(X_t) = \sigma^2. \tag{3.6}$$

3.1.3 Definition White noise*: sequence of* $r.v.'s$ *in* L_2 *of mean zero, of same variance and mutually uncorrelated,* $i.e.$

$$E(X_t^2) < \infty, \forall t \in T, \tag{3.7}$$

$$E(X_t^2) < \infty, \forall t \in T, \tag{3.8}$$

$$E(X_t^2) = \sigma^2 \ , \forall t \in T, \tag{3.9}$$

$$Cov(X_s, X_t) = 0\ , \ \text{if}\ s \neq t. \tag{3.10}$$

We write :

$$\{X_t : t \in T\} \sim BB(0, \sigma^2) \text{ or } \{X_t\} \sim BB(0, \sigma^2). \tag{3.11}$$

3.1.4 Definition Heteroskedastic white noise*: sequence of* $r.v.'s$ *in* L_2 *with mean zero and mutually uncorrelated, i.e.*

$$E(X_t^2) < \infty, \forall t \in T, \tag{3.12}$$

$$E(X_t) = 0, \forall t \in T, \tag{3.13}$$

$$Cov(X_t, X_s) = 0,\ \text{if}\ s \neq t, \tag{3.14}$$

$$E(X_t^2) = \sigma_t^2\ ,\ \forall t \in T. \tag{3.15}$$

We write:

$$\{X_t : t \in \mathbb{Z}\} \ \sim \ BB(0, \sigma_t^2) \text{ or } \{X_t\} \ \sim \ BB(0, \sigma_t^2). \tag{3.16}$$

Each one of these four models will be called a *noise* process.

3.2. Harmonic processes

Many time series exhibit apparent periodic behavior. This suggests one to use periodic functions to describe them.

3.2.1 Definition *A function* $f(t)$, $t \in \mathbb{R}$, *is periodic of period* P *if*

$$f(t + P) = f(t), \forall t.$$

$\frac{1}{P}$ *is the frequency associated with the function (number of cycles per unit of time).*

3.2.2 Example

$$\sin(t) = \sin(t + 2\pi) = \sin(t + 2\pi k), \forall k \in \mathbb{Z}. \tag{3.17}$$

3.2.3 Example

$$\cos(t) = \cos(t + 2\pi) = \cos(t + 2\pi k), \forall k \in \mathbb{Z}. \tag{3.18}$$

3.2.4 Example

$$\sin(\nu t) = \sin\left[\nu\left(t + \frac{2\pi}{v}\right)\right] = \sin\left[\nu\left(t + \frac{2\pi k}{v}\right)\right]\ , \forall k \in \mathbb{Z}. \tag{3.19}$$

3.2.5 Example

$$\cos(\nu t) = \cos\left[\nu\left(t + \frac{2\pi}{v}\right)\right] = \cos\left[\nu\left(t + \frac{2\pi k}{v}\right)\right]\ , \forall k \in \mathbb{Z}. \tag{3.20}$$

For $\sin(\nu t)$ and $\cos(\nu t)$, the period is $P = 2\pi/\nu$.

3.2.6 Example

$$\begin{aligned} f(t) &= C\ \cos(\nu t + \theta) = C[\cos(\nu t)\cos(\theta) - \sin(\nu t)\sin(\theta)] \\ &= A\ \cos(\nu t) + B\ \sin(\nu t) \end{aligned} \tag{3.21}$$

where $C \geq 0$, $A = C\ \cos(\theta)$ and $B = -C\ \sin\ \theta$. Further,

$$C = \sqrt{A^2 + B^2}\,, \quad \tan(\theta) = -B/A \text{ (if } C \neq 0). \tag{3.22}$$

3.2.7 Definition *We call:*

$$\begin{aligned} C &= \textit{amplitude;} \\ \nu &= \textit{angular mfrequency (radians/time unit);} \\ P &= 2\pi/\nu = \textit{period;} \\ \bar{v} &= \frac{1}{P} = \frac{v}{2\pi} = \textit{ frequency (number of cycles per time unit);} \\ \theta &= \textit{phase angle (usually } 0 \leq \theta < 2\pi \textit{ or } -\pi/2 < \theta \leq \pi/2). \end{aligned}$$

3.2.8 Example

$$f(t) = C\ \sin(\nu t + \theta) = C\ \cos(\nu t + \theta - \pi/2) \tag{3.23}$$

$$= C[\sin(\nu t)\cos(\theta) + \cos(\nu t)\sin(\theta)] \tag{3.24}$$

$$= A\ \cos(\nu t) + B\ \sin(\nu t) \tag{3.25}$$

where

$$0 \leq \nu < 2\pi\,, \tag{3.26}$$

$$A = C\ \sin(\theta) = C\ \cos\left(\theta - \frac{\pi}{2}\right)\,, \tag{3.27}$$

$$B = C\ \cos(\theta) = -C\ \sin\left(\theta - \frac{\pi}{2}\right)\ . \tag{3.28}$$

Consider the model

$$\begin{aligned} X_t &= C\ \cos(\nu t + \theta) & (3.29)\\ &= A\ \cos(\nu t) + B\ \sin(\nu t), t \in \mathbb{Z}. & (3.30)\end{aligned}$$

If A and B are constants,

$$E(X_t) = A\ \cos(\nu t) + B\ \sin(\nu t)\ ,\ t \in \mathbb{Z}, \tag{3.31}$$

and thus the process X_t is non-stationary (the mean is not constant). Suppose now A and B are $r.v.'s$ such that

$$E(A) = E(B) = 0, E(A^2) = E(B^2) = \sigma^2, E(AB) = 0. \tag{3.32}$$

A and B do not depend on t but are fixed for each realization of the process $[A = A(\omega)$, $B = B(\omega)]$. In this case,

$$\begin{aligned} E(X_t) &= 0, & (3.33)\\ E(X_s X_t) &= E(A^2)\cos(\nu s)\cos(\nu t) + E(B^2)\sin(\nu s)\sin(\nu t) \\ &= \sigma^2[\cos(\nu s)\cos(\nu t) + \sin(\nu s)\sin(\nu t)] = \sigma^2\cos[\nu(t-s)]. & (3.34)\end{aligned}$$

The process X_t is stationary of order 2 with the following autocovariance and autocorrelation functions:

$$\gamma_X(k) = \sigma^2\cos(\nu k), \rho_X(k) = \cos(\nu k). \tag{3.35}$$

If we add m cyclic processes of the form (3.29), we obtain a harmonic process of order m.

3.2.9 Definition *(Harmonic process of order m). We say the process $\{X_t : t \in T\}$ is a harmonic process of order m if it can written in the form*

$$X_t = \sum_{j=1}^{m}[A_j\cos(\nu_j t) + B_j\sin(\nu_j t)],\ \forall t \in T\,, \tag{3.36}$$

where ν_1, ... , ν_m are distinct constants in the interval $[0,\ 2\pi)$.

If we suppose A_j, B_j , $j = 1, \ldots, m$, are $r.v.'s$ in L_2 such that

$$
\begin{aligned}
E(A_j) &= E(B_j) = 0, E(A_j^2) = E(B_j^2) = \sigma_j^2, j = 1, \ldots, m , && (3.37)\\
E(A_j A_k) &= E(B_j B_k) = 0, pour j \neq k, && (3.38)\\
E(A_j B_k) &= 0, \forall j, k , && (3.39)
\end{aligned}
$$

the process X_t can be considered second-order stationary:

$$
\begin{aligned}
E(X_t) &= 0 , && (3.40)\\
E(X_s X_t) &= \sum_{j=1}^{m} \sigma_j^2 \cos[\nu_j (t-s)] , && (3.41)
\end{aligned}
$$

hence

$$\gamma_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) , \tag{3.42}$$

$$\rho_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) / \sum_{j=1}^{m} \sigma_j^2 . \tag{3.43}$$

If we add a white noise u_t to X_t in (3.36), we obtain again a second-order stationary process :

$$X_t = \sum_{j=1}^{m} [A_j \cos(\nu_j t) + B_j \sin(\nu_j t)] + u_t, t \in T , \tag{3.44}$$

where the process $\{u_t : t \in T\} \sim BB(0, \sigma^2)$ is uncorrelated with A_j, B_j , $j = 1, \ldots, m$. In this case, $E(X_t) = 0$ and

$$\gamma_X(k) = \sum_{j=1}^{m} \sigma_j^2 \cos(\nu_j k) + \sigma^2 \delta(k) \tag{3.45}$$

where $\delta(k) = 1$ for $k = 0$, and $\delta(k) = 0$ otherwise. If a series can be described by an equation of the form (3.44), we can view it as a realization of a second-order stationary process.

3.3. Linear processes

Many stochastic processes with dependence are obtained as transformations of noise processes.

3.3.1 Definition *The process* $\{X_t : t \in T\}$ *is an autoregressive process of order* p if *it satisfies and equation of the form*

$$X_t = \bar{\mu} \; + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t \, , \forall t \in T \, , \tag{3.46}$$

where $\{u_t : t \in \mathbb{Z}\} \; \sim BB(0, \sigma^2)$*. In this case, we denote*

$$\{X_t : t \in T\} \; \sim AR(p).$$

Usually, $T = \mathbb{Z}$ *or* $T = \mathbb{Z}_+$ *(positive integers). If* $\sum_{j=1}^{p} \varphi_j \neq 1$*, we can define* $\mu = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j)$ *and write*

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t, \forall t \in T,$$

where $\tilde{X}_t \equiv X_t - \mu$*.*

3.3.2 3.3.3 Definition *The process* $\{X_t : t \in T\}$ *is a moving average process of order* q if *it can written in the form*

$$X_t = \bar{\mu} \; + \sum_{j=0}^{q} \psi_j u_{t-j}, \forall t \in T, \tag{3.47}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$*. In this case, we denote*

$$\{X_t : t \in T\} \sim \; MA(q). \tag{3.48}$$

Without loss of generality, we can set $\psi_0 = 1$ and $\psi_j = -\theta_j$, $j = 1, ... , q$:

$$X_t = \bar{\mu} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \, , t \in T$$

or, equivalently,

$$\tilde{X}_t = u_t - \sum_{j=1}^{q} \theta_j u_{t-j}$$

where $\tilde{X}_t \equiv X_t - \bar{\mu}$.

3.3.4 Definition *The process* $\{X_t : t \in T\}$ *is an autoregressive-moving-average (ARMA) process of order* (p, q) *if it can be written in the form*

$$X_t = \bar{\mu} \ + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}, \forall t \in T, \tag{3.49}$$

where $\{u_t : t \in \mathbb{Z}\} \ \sim \ BB(0, \sigma^2)$*. In this case, we denote*

$$\{X_t : t \in T\} \ \sim ARMA(p, q). \tag{3.50}$$

If $\sum_{j=1}^{p} \varphi_j \neq 1$, we can also write

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{3.51}$$

where $\tilde{X}_t = X_t - \mu$ and $\mu = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j)$.

3.3.5 Definition *The process* $\{X_t : t \in T\}$ *is a moving-average process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} \ + \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z}, \tag{3.52}$$

where $\{u_t : t \in \mathbb{Z}\} \ \sim \ BB(0, \sigma^2)$ *. We also say that* X_t *is a weakly linear process. In this case, we denote*

$$\{X_t : t \in T\} \ \sim \ MA(\infty). \tag{3.53}$$

In particular, if $\psi_j = 0$ *for* $j < 0$, $i.e.$

$$X_t = \bar{\mu} + \sum_{j=0}^{\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z}, \tag{3.54}$$

we say that X_t *is a causal function of* u_t *(causal linear process). [Box and Jenkins (1976) speak about general linear processes.]*

3.3.6 Definition *The process* $\{X_t : t \in T\}$ *is an autoregressive process of infinite order if it can be written in the form*

$$X_t = \bar{\mu} + \sum_{j=1}^{\infty} \varphi_j X_{t-j} + u_t, t \in T, \tag{3.55}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$ *. In this case, we denote*

$$\{X_t : t \in T\} \sim AR(\infty). \tag{3.56}$$

3.3.7 Remark *Generalization:* We can generalize the notions defined above by assuming that $\{u_t : t \in \mathbb{Z}\}$ is a noise. Unless sated otherwise, we will suppose $\{u_t\}$ is a white noise.

3.3.8 QUESTIONS :

1. Under which conditions are the processes defined above stationary (strictly or in L_r)?
2. Under which conditions are the processus $MA(\infty)$ or $AR(\infty)$ well defined (convergent series)?
3. What are the links between the different classes of processes defined above?
4. When a process is stationary, what are its autocovariance and autocorrelation functions?

3.4. Integrated processes

3.4.1 Definition *The process* $\{X_t : t \in T\}$ *is a random walk if it satisfies an equation of the form*

$$X_t - X_{t-1} = v_t, \forall t \in T, \tag{3.57}$$

where $\{v_t : t \in \mathbb{Z}\} \sim$ *IID. For such a process to be well defined, we must suppose that* $n_0 \neq -\infty$ *(the process ne can start at* $-\infty$*). If* $n_0 = -1$*, we can write*

$$X_t = X_0 + \sum_{j=1}^{t} v_j \tag{3.58}$$

hence the name "integrated process". If $E(v_t) = \bar{\mu}$ or $Med(v_t) = \bar{\mu}$, one often writes

$$X_t - X_{t-1} = \bar{\mu} + u_t \tag{3.59}$$

where $u_t \equiv v_t - \bar{\mu} \sim$ IID and $E(u_t) = 0$ or $Med(u_t) = 0$ (depending on whether $E(u_t) = 0$ or $Med(u_t) = 0$). If $\bar{\mu} \neq 0$, the random walk has drift.

3.4.2 Definition *The process $\{X_t : t \in T\}$ is a random walk generated by a white noise [or an heteroskedastic white noise, or a sequence of independent $r.v.'s$] If X_t satisfies an equation of the form*

$$X_t - X_{t-1} = \bar{\mu} + u_t \tag{3.60}$$

where $\{u_t : t \in T\} \sim BB(0, \sigma^2)$ [or $\{u_t : t \in T\} \sim BB(0, \sigma_t^2)$, or $\{u_t : t \in T\} \sim IND(0)$].

3.4.3 Definition *The process $\{X_t : t \in T\}$ is* integrated *of order d if it can be written in the form*

$$(1 - B)^d X_t = Z_t \, , \forall t \in T, \tag{3.61}$$

where $\{Z_t : t \in T\}$ is a stationary process (usually stationary of order 2) and d is a non-negative integer $(d = 0, 1, 2, ...)$. In particular, if $\{Z_t : t \in T\}$ is an $ARMA(p, q)$ stationary process, $\{X_t : t \in T\}$ is an $ARIMA(p, d, q)$ process: $\{X_t : t \in T\} \sim ARIMA(p, d, q)$. We note

$$\begin{aligned}
B\, X_t &= X_{t-1} \, , && (3.62)\\
(1 - B)X_t &= X_t - X_{t-1} \, , && (3.63)\\
(1 - B)^2 X_t &= (1 - B)(1 - B)X_t = (1 - B)(X_t - X_{t-1}) && (3.64)\\
&= X_t - 2X_{t-1} + X_{t-2}, && (3.65)\\
(1 - B)^d X_t &= (1 - B)(1 - B)^{d-1} X_t, d = 1, 2, ... && (3.66)
\end{aligned}$$

where $(1 - B)^0 = 1$.

3.5. Models of deterministic tendency

3.5.1 Definition *The process $\{X_t : t \in T\}$ follows a deterministic tendency if it can be written in the form*

$$X_t = f(t) + Z_t \, , \forall t \in T, \tag{3.67}$$

where $f(t)$ is a deterministic function of time and $\{Z_t : t \in T\}$ is a noise or a stationary process.

3.5.2 Important cases of deterministic tendency:

$$X_t = \beta_0 + \beta_1 t + u_t, \tag{3.68}$$

$$X_t = \sum_{j=0}^{k} \beta_j t^j + u_t, \tag{3.69}$$

where $\{u_t : t \in T\} \sim BB(0, \sigma^2)$.

4. Transformations of stationary processes

4.1 Theorem *Let $\{X_t : t \in \mathbb{Z}\}$ be a stochastic process on the integers, $r \geq 1$ and $\{a_j : j \in \mathbb{Z}\}$ a sequence of real numbers. If $\sum_{j=-\infty}^{\infty} |a_j| E(|X_{t-j}|^r)^{1/r} < \infty$, then, for any t, the random series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely $a.s.$ and in mean of order r to a $r.v.$ Y_t such that $E(|Y_t|^r) < \infty$.*

PROOF: See Dufour (1999a).

4.2 Theorem *Let $\{X_t : t \in \mathbb{Z}\}$ be a second-order stationary process and $\{a_j : j \in \mathbb{Z}\}$ an sequence of real numbers absolutely convergent sequence of real numbers, $i.e.$ $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then the random series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely $p.s.$ and in mean of order 2 to a $r.v.$ $Y_t \in L_2, \forall t$, and the process $\{Y_t : t \in \mathbb{Z}\}$ is second-order stationary.*

PROOF : See Gouriéroux and Monfort (1997, Property 5.6).

4.3 Theorem *If $\{X_t : t \in \mathbb{Z}\}$ be a second-order stationary process with autocovariance function $\gamma_X(k)$, the autocovariance function of the transformed process*

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}, \tag{4.1}$$

where $\sum_{j=-\infty}^{\infty} |a_j| < \infty$, is given by

$$\gamma_Y(k) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \gamma_X(k-i+j) \; . \tag{4.2}$$

4.4 Theorem *The series $\sum_{j=-\infty}^{\infty} a_j X_{t-j}$ converges absolutely $p.s.$ for any second-order stationary process $\{X_t : t \in \mathbb{Z}\}$ iff*

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty. \tag{4.3}$$

5. Infinite order moving averages

Consider the random series

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, t \in \mathbb{Z} \tag{5.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim \; BB(0, \sigma^2)$.

5.1. Convergence conditions

We can write

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = \sum_{j=-\infty}^{\infty} Y_j(t) = \sum_{j=-\infty}^{-1} Y_j(t) + \sum_{j=0}^{\infty} Y_j(t) \tag{5.2}$$

where $Y_j(t) \equiv \psi_j u_{t-j}$ and

$$E[|Y_j(t)|] = |\psi_j| E[|u_{t-j}|] \leq |\psi_j| [E(u_{t-j}^2)]^{\frac{1}{2}} = |\psi_j| \sigma < \infty,$$

$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ is a series of orthogonal variables.

Suppose $\sum_{j=-\infty}^{-1} \psi_j^2 < \infty$. Then

$$Y_m^1(t) \equiv \sum_{j=-m}^{-1} \psi_j u_{t-j} \underset{m\to\infty}{\overset{2}{\to}} Y^1(t) \equiv \sum_{j=-\infty}^{-1} \psi_j u_{t-j},$$

$$Y_n^2(t) \equiv \sum_{j=0}^{n} \psi_j u_{t-j} \underset{n\to\infty}{\overset{2}{\to}} Y^2(t) \equiv \sum_{j=1}^{\infty} \psi_j u_{t-j}$$

[see Dufour (1999a)], and thus

$$Y_{m,n}(t) \equiv Y_m^1(t) + Y_n^2(t) \underset{\substack{m\to\infty \\ n\to\infty}}{\overset{2}{\to}} \tilde{X}_t \equiv Y^1(t) + Y^2(t) \equiv \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}, \forall t \in \mathbb{Z}.$$

It is also clear that

$$X_n(t) \equiv Y_n^1(t) + Y_n^2(t) = \sum_{j=-n}^{-1} \psi_j u_{t-j} + \sum_{j=0}^{n} \psi_j u_{t-j} \underset{n\to\infty}{\overset{2}{\to}} \tilde{X}_t \equiv \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}\,,\ \forall t \in \mathbb{Z}\,. \tag{5.3}$$

Thus,

$$\sum_{j=-\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a } r.v.\ \tilde{X}_t$$

[see Dufour (1999a)]. Further

$$\sum_{j=-\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a } r.v.\ \tilde{X}_t$$

[see Dufour (1999a)],

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |\psi_j| &< \infty \Rightarrow \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty \\ &\Rightarrow \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \text{ converges in } q.m. \text{ to a } \tilde{X}_t. \end{aligned}$$

If the variables $\{u_t : t \in \mathbb{Z}\}$ are mutually independent,

$$\sum_{j=-\infty}^{+\infty} \psi_j^2 < \infty \Rightarrow \sum_{j=-\infty}^{+\infty} \psi_j u_{t-j} \text{ converges in } a.s. \text{ to a } r.v.\ \tilde{X}_t$$

[see Dufour (1999a)]. The variable $\tilde{X}_t$ is called the limit (in $q.m.$ or $a.s.$) of the series $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$, and we write

$$\tilde{X}_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}.$$

on defining $X_t \equiv \mu + \tilde{X}_t$, we obtain the linear process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$$

where it is assumed that the series converges.

5.2. Mean, variance and covariances

By (5.3), we have:

$$\begin{aligned} &E[X_n(t)] \underset{n\to\infty}{\rightarrow} E(\tilde{X}_t)\,, \\ &E[X_n(t)^2] \underset{n\to\infty}{\rightarrow} E(\tilde{X}_t^2), \\ &E[X_n(t)X_n(t+k)] \underset{n\to\infty}{\rightarrow} E(\tilde{X}_t\,\tilde{X}_{t+k}); \end{aligned}$$

see Dufour (1999a). Consequently,

$$E(\tilde{X}_t) = 0\,, \tag{5.4}$$

$$Var(\tilde{X}_t) = E(\tilde{X}_t^2) = \lim_{n\to\infty} \sum_{j=-n}^{n} \psi_j^2 \sigma^2 = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j^2\,, \tag{5.5}$$

$$\begin{aligned} Cov(\tilde{X}_t, \tilde{X}_{t+k}) &= E(\tilde{X}_t\,\tilde{X}_{t+k}) \\ &= \lim_{n\to\infty} E\left[\left(\sum_{i=-n}^{n} \psi_i u_{t-i}\right)\left(\sum_{j=-n}^{n} \psi_j u_{t+k-j}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=-n}^{n} \sum_{j=-n}^{n} \psi_i \psi_j E(u_{t-i}u_{t+k-j}) \\
&= \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{i=-n}^{n-k} \psi_i \psi_{i+k}\sigma^2 = \sigma^2 \sum\limits_{i=-\infty}^{\infty} \psi_i\psi_{i+k}, & \text{if } k \geq 1, \\ \lim\limits_{n\to\infty} \sum\limits_{j=-n}^{n} \psi_j \psi_{j+|k|}\sigma^2 = \sigma^2 \sum\limits_{j=-\infty}^{\infty} \psi_j\psi_{j+|k|}\,, & \text{if } k \leq -1, \end{cases}
\end{aligned} \tag{5.6}
$$

since $t-i = t+k-j \Rightarrow j = i+k$ and $i = j-k$. For any $k \in \mathbb{Z}$, we can write

$$
Cov(\tilde{X}_t, \tilde{X}_{t+k}) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}\,, \tag{5.7}
$$

$$
Corr(\tilde{X}_t, \tilde{X}_{t+k}) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=-\infty}^{\infty} \psi_j^2\,. \tag{5.8}
$$

The series $\sum\limits_{j=-\infty}^{\infty} \psi_j \psi_{j+k}$ converges absolutely, for

$$
\left| \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \right| \leq \sum_{j=-\infty}^{\infty} |\psi_j \psi_{j+k}| \leq \left[\sum_{j=-\infty}^{\infty} \psi_j^2 \right]^{\frac{1}{2}} \left[\sum_{j=-\infty}^{\infty} \psi_{j+k}^2 \right]^{\frac{1}{2}} < \infty\,. \tag{5.9}
$$

If $X_t = \mu + \tilde{X}_t = \mu + \sum\limits_{j=-\infty}^{+\infty} \psi_j u_{t-j}$, then

$$
E(X_t) = \mu\,,\ Cov(X_t, X_{t+k}) = Cov(\tilde{X}_t, \tilde{X}_{t+k}). \tag{5.10}
$$

In the case of a causal $MA(\infty)$ process causal, we have

$$
X_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \tag{5.11}
$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$,

$$
Cov(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}\,, \tag{5.12}
$$

$$
Corr(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=0}^{\infty} \psi_j^2\,. \tag{5.13}
$$

5.3. Stationarity

The process

$$X_t = \mu \, + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \, , t \in \mathbb{Z}, \tag{5.14}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, is second-order stationary, for $E(X_t)$ and $Cov(X_t, X_{t+k})$ do not depend on t. If we suppose that $\{u_t : t \in \mathbb{Z}\} \sim$ IID, with $E|u_t| < \infty$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, the process is strictly stationary.

5.4. Operational notation

We can denote the process $MA(\infty)$

$$X_t = \mu + \psi(B)u_t = \mu + \left(\sum_{j=-\infty}^{\infty} \psi_j B^j \right) u_t \tag{5.15}$$

where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$ and $B^j u_t = u_{t-j}$.

6. Finite order moving averages

6.1 The $MA(q)$ process can be written

$$X_t = \mu + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{6.1}$$

where $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$. This process is a special case of the $MA(\infty)$ process with

$$\begin{aligned} \psi_0 &= 1 \, , \psi_j = -\theta_j \, , \text{ for } 1 \le j \le q \, , \\ \psi_j &= 0 \, , \text{ for } j < 0 \text{ or } j > q. \end{aligned} \tag{6.2}$$

6.2 This process is clearly second-order stationary, with

$$E(X_t) = \mu \, , \tag{6.3}$$

$$V(X_t) \;=\; \sigma^2\left(1+\sum_{j=1}^{q}\theta_j^2\right), \tag{6.4}$$

$$\gamma(k) \;\equiv\; Cov(X_t, X_{t+k}) = \sigma^2\sum_{j=-\infty}^{\infty}\psi_j\psi_{j+|k|}\,. \tag{6.5}$$

On defining $\theta_0 \equiv -1$, we then see that

$$\begin{aligned}
\gamma(k) &= \sigma^2\sum_{j=0}^{q-k}\theta_j\theta_{j+k} \\
&= \sigma^2\left[-\theta_k+\sum_{j=1}^{q-k}\theta_j\theta_{j+k}\right] \\
&= \sigma^2[-\theta_k+\theta_1\theta_{k+1}+...+\theta_{q-k}\theta_q]\,,\ \text{for } 1\le k\le q, && (6.6)\\
\gamma(k) &= 0\,,\ \text{for } k\ge q+1, \\
\gamma(-k) &= \gamma(k)\,,\ \text{for } k<0. && (6.7)
\end{aligned}$$

The autocorrelation function of X_t is thus

$$\begin{aligned}
\rho(k) &= \left(-\theta_k+\sum_{j=1}^{q-k}\theta_j\theta_{j+k}\right)\Big/\left(1+\sum_{j=1}^{q}\theta_j^2\right), \quad 1\le k\le q \\
&= 0\,, \qquad\qquad k\ge q+1
\end{aligned} \tag{6.8}$$

The autocorrelations are zero for $k \ge q+1$.

6.3 For $q = 1$,

$$\begin{aligned}
\rho(k) &= -\theta_1/(1+\theta_1^2), & k=1\,, \\
&= 0\,, & k\ge 2,
\end{aligned} \tag{6.9}$$

hence $|\rho(1)| \le 0.5$.

6.4 For $q = 2$,

$$\begin{aligned}
\rho(k) &= (-\theta_1+\theta_1\theta_2)/(1+\theta_1^2+\theta_2^2)\,, & k=1\,, \\
&= -\theta_2/(1+\theta_1^2+\theta_2^2)\,, & k=2\,, \\
&= 0\,, & k\ge 3\,,
\end{aligned} \tag{6.10}$$

hence $|\rho(2)| \le 0.5$.

6.5 For any $MA(q)$ process,

$$\rho(q) = -\theta_q/(1 + \theta_1^2 + ... + \theta_q^2)\,, \tag{6.11}$$

hence $|\rho(q)| \leq 0.5$.

6.6 There are general constraints on the autocorrelations of an $MA(q)$ process:

$$|\rho(k)| \leq \cos(\pi/\{[q/k] + 2\}) \tag{6.12}$$

where $[x]$ is the largest integer less than or equal to x. From the latter formula, we see:

$$\begin{array}{ll} \text{for } q = 1\,, & |\rho(1)| \leq \cos(\pi/3) = 0.5, \\ \text{for } q = 2\,, & |\rho(1)| \leq \cos(\pi/4) = 0.7071, \\ & |\rho(2)| \leq \cos(\pi/3) = 0.5, \\ \text{for } q = 3\,, & |\rho(1)| \leq \cos(\pi/5) = 0.809, \\ & |\rho(2)| \leq \cos(\pi/3) = 0.5, \\ & |\rho(3)| \leq \cos(\pi/3) = 0.5. \end{array} \tag{6.13}$$

See Chanda (1962), and Kendall, Stuart, and Ord (1983, p. 519).

7. Autoregressive processes

7.1 Consider a process $\{X_t : t \in \mathbb{Z}\}$ which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t, \forall t \in \mathbb{Z}, \tag{7.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. In symbolic notation,

$$\varphi(B)X_t = \bar{\mu} + u_t, t \in \mathbb{Z}, \tag{7.2}$$

where $\varphi(B) = 1 - \varphi_1 B - ... - \varphi_p B^p$.

7.2 Stationarity

Consider the process AR(1)

$$X_t = \varphi_1 X_{t-1} + u_t, \varphi_1 \neq 0. \tag{7.3}$$

If X_t is S2 ,

$$E(X_t) = \varphi_1 E(X_{t-1}) = \varphi_1 E(X_t), \tag{7.4}$$

hence $E(X_t) = 0$. By successive substitutions,

$$\begin{aligned} X_t &= \varphi_1[\varphi_1 X_{t-2} + u_{t-1}] + u_t \\ &= u_t + \varphi_1 u_{t-1} + \varphi_1^2 X_{t-2} \\ &= \sum_{j=0}^{N-1} \varphi_1^j u_{t-j} + \varphi_1^N X_{t-N} \ . \end{aligned} \tag{7.5}$$

If we suppose that X_t is S2 with $E(X_t^2) \neq 0$, we see that

$$E\left[\left(X_t - \sum_{j=0}^{N-1} \varphi_1^j u_{t-j}\right)^2\right] = \varphi_1^{2N} E(X_{t-N}^2) = \varphi_1^{2N} E(X_t^2) \underset{N\to\infty}{\to} 0 \Leftrightarrow |\varphi_1| < 1. \tag{7.6}$$

The series $\sum_{j=0}^{\infty} \varphi_1^j u_{t-j}$ converges in $q.m.$ to X_t :

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} \equiv (1 - \varphi_1 B)^{-1} u_t = \frac{1}{1 - \varphi_1 B} \ u_t \tag{7.7}$$

where

$$(1 - \varphi_1 B)^{-1} = \sum_{j=0}^{\infty} \varphi_1^j B^j. \tag{7.8}$$

Since

$$\sum_{j=0}^{\infty} E|\varphi_1^j u_{t-j}| \leq \sigma \sum_{j=0}^{\infty} |\varphi_1|^j = \frac{\sigma}{1 - |\varphi_1|} < \infty \tag{7.9}$$

when $|\varphi_1| < 1$, the convergence is also $a.s.$ The process $X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j}$ is S2.

When $|\varphi_1| < 1$, the difference equation

$$(1 - \varphi_1 B) X_t = u_t \tag{7.10}$$

has a unique stationary solution which can be written

$$X_t = \sum_{j=0}^{\infty} \varphi_1^j u_{t-j} = (1 - \varphi_1 B)^{-1} u_t. \tag{7.11}$$

The latter is thus a causal $MA(\infty)$ process.

This condition is sufficient (but non necessary) for the existence of a unique stationary solution. The stationarity condition is often expressed by saying that the polynome $\varphi(z) = 1 - \varphi_1 z$ has all its roots outside the unit circle $|z| = 1$:

$$1 - \varphi_1 z_* = 0 \Leftrightarrow z_* = \frac{1}{\varphi_1}, \tag{7.12}$$

where $|z_*| = 1/|\varphi_1| > 1$. In this case, we also have $E(X_{t-k} u_t) = 0, \forall k \geq 1$. The same conclusion holds if we consider the general process

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t \, . \tag{7.13}$$

For the $AR(p)$ process,

$$X_t = \bar{\mu} \ + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t \tag{7.14}$$

or

$$\varphi(B) X_t = \bar{\mu} + u_t, \tag{7.15}$$

the stationarity condition is the following :

$$\text{if the polynome } \varphi(z) = 1 - \varphi_1 z - ... - \varphi_p z^p \text{ has all its roots outside the unit circle,} \\ \text{the equation (7.14) has one and only one weakly statiinary solution.} \tag{7.16}$$

The order p polynome $\varphi(z)$ can be written

$$\varphi(z) = (1 - G_1 z)(1 - G_2 z)...(1 - G_p z) \tag{7.17}$$

and has the roots

$$z_1^* = 1/G_1, ..., z_p^* = 1/G_p. \tag{7.18}$$

The stationarity condition may then be written:

$$|G_j| < 1, j = 1, ..., p. \tag{7.19}$$

The solution stationary can be written

$$X_t = \varphi(B)^{-1}\bar{\mu} + \varphi(B)^{-1}u_t = \mu + \varphi(B)^{-1}u_t \tag{7.20}$$

where

$$\mu = \bar{\mu}/(1 - \sum_{j=1}^{p}\varphi_j), \tag{7.21}$$

$$\begin{aligned} \varphi(B)^{-1} &= \mathop{\mathit{\Pi}}_{j=1}^{p}(1 - G_j B)^{-1} = \mathop{\mathit{\Pi}}_{j=1}^{p}\left(\sum_{k=0}^{\infty} G_j^k B^k\right) \\ &= \sum_{j=1}^{p}\frac{K_j}{1 - G_j B} \end{aligned} \tag{7.22}$$

and K_1, ... , K_p are constants (expansion in partial fractions). Consequently,

$$\begin{aligned} X_t &= \mu + \sum_{j=1}^{p}\frac{K_j}{1 - G_j B}\, u_t \\ &= \mu + \sum_{k=0}^{\infty}\psi_k u_{t-k} = \mu + \psi(B)u_t \end{aligned} \tag{7.23}$$

where $\psi_k = \sum_{j=1}^{p} K_j G_j^k$. Thus

$$E(X_{t-j}u_t) = 0, \forall j \geq 1. \tag{7.24}$$

For the process AR(1) and AR(2), the stationarity conditions can be written as follows.

(a) AR(1) : $(1 - \varphi_1 B)X_t = \bar{\mu} + u_t$

$$|\varphi_1| < 1 \tag{7.25}$$

(b) AR(2) : $(1 - \varphi_1 B - \varphi_2 B^2)X_t = \bar{\mu} + u_t$

$$\varphi_2 + \varphi_1 < 1 \tag{7.26}$$

$$\varphi_2 - \varphi_1 < 1 \tag{7.27}$$

$$-1 < \varphi_2 < 1 \tag{7.28}$$

7.3 Mean, variance and autocovariances

Suppose:

a) the autoregressive process X_t is second-order stationary with $\sum_{j=1}^{p}\varphi_j \neq 1$
and
b) $E(X_{t-j}u_t) = 0\ , \forall j \geq 1\ ,$ (7.29)

i.e. we assume X_t is a weakly stationary solution of the equation (7.14) such that $E(X_{t-j}u_t) = 0, \forall j \geq 1$.

By the stationarity assumption,

$$E(X_t) = \mu, \forall t \Rightarrow \mu = \bar{\mu}\ + \sum_{j=1}^{p}\varphi_j\mu \Rightarrow E(X_t) = \mu = \bar{\mu}/\left(1 - \sum_{j=1}^{p}\varphi_j\right) \tag{7.30}$$

For stationarity to hold, it is necessary that $\sum_{j=1}^{p}\varphi_j \neq 1$. Let us rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p}\varphi_j\tilde{X}_{t-j} + u_t \tag{7.31}$$

where $\tilde{X}_t = X_t - \mu\ , E(\tilde{X}_t) = 0$. Then, for $k \geq 0$,

$$\begin{aligned}
\tilde{X}_{t+k} &= \sum_{j=1}^{p}\varphi_j\tilde{X}_{t+k-j} + u_{t+k}, && (7.32)\\
E(\tilde{X}_{t+k}\ \tilde{X}_t) &= \sum_{j=1}^{p}\varphi_j E(\tilde{X}_{t+k-j}\ \tilde{X}_t) + E(u_{t+k}\ \tilde{X}_t), && (7.33)\\
\gamma(k) &= \sum_{j=1}^{p}\varphi_j\gamma(k-j) + E(u_{t+k}\ \tilde{X}_t), && (7.34)
\end{aligned}$$

where

$$\begin{aligned}
E(u_{t+k}\ \tilde{X}_t) &= \sigma^2, \quad \text{if } k = 0,\\
&= 0\ , \quad \text{if } k \geq 1.
\end{aligned} \tag{7.35}$$

Thus

$$\rho(k) = \sum_{j=1}^{p}\varphi_j\rho(k-j), k \geq 1. \tag{7.36}$$

These formulae are called the "Yule-Walker equations". If we know $\rho(0), \dots, \rho(p-1)$, we can easily compute $\rho(k)$ for $k \geq p+1$. We can also write the Yule-Walker equations in the form:

$$\varphi(B)\rho(k) = 0, k \geq 1, \tag{7.37}$$

where $B^j\rho(k) \equiv \rho(k-j)$. To obtain $\rho(1), \, ... \, , \rho(p-1)$ when $p > 1$, it is sufficient to solve the linear equation system:

$$\begin{aligned}
\rho(1) &= \varphi_1 + \varphi_2\rho(1) + ... + \varphi_p\rho(p-1) \\
\rho(2) &= \varphi_1\rho(1) + \varphi_2 + ... + \varphi_p\rho(p-2) \\
&\vdots \\
\rho(p-1) &= \varphi_1\rho(p-2) + \varphi_2\rho(p-3) + ... + \varphi_p\rho(1)
\end{aligned} \tag{7.38}$$

where we use the identity $\rho(-j) = \rho(j)$. The other autocorrelations may then be obtained by recurrence:

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j), k \geq p. \tag{7.39}$$

To compute $\gamma(0) = Var(X_t)$, we solve the equation

$$\begin{aligned}
\gamma(0) &= \sum_{j=1}^{p} \varphi_j \gamma(-j) + E(u_t \, \tilde{X}_t) \\
&= \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2,
\end{aligned} \tag{7.40}$$

hence, using $\gamma(j) = \rho(j)\gamma(0)$,

$$\gamma(0) \left[1 - \sum_{j=1}^{p} \varphi_j \rho(j) \right] = \sigma^2 \tag{7.41}$$

and

$$\gamma(0) = \frac{\sigma^2}{1 - \sum\limits_{j=1}^{p} \varphi_j \rho\,(j)}. \tag{7.42}$$

7.4 Special cases

1. AR(1) : $\tilde{X}_t = \varphi_1 \, \tilde{X}_{t-1} + u_t$

$$\rho(1) = \varphi_1 \tag{7.43}$$

$$\begin{aligned}\rho(k) &= \varphi_1\rho(k-1)\,,\ k \geq 1 && (7.44)\\ \rho(2) &= \varphi_1\rho(1) = \varphi_1^2 && (7.45)\\ \rho(k) &= \varphi_1^k, k \geq 1 && (7.46)\\ \gamma(0) &= Var(X_t) = \frac{\sigma^2}{1-\varphi_1^2} && (7.47)\end{aligned}$$

These is no constraint on $\rho(1)$, but there are constraints on $\rho(k)$ for $k \geq 2$.

2. AR(2) : $X_t = \varphi_1\tilde{X}_{t-1} + \varphi_2\tilde{X}_{t-2} + u_t$

$$\begin{aligned}\rho(1) &= \varphi_1 + \varphi_2\rho(1) && (7.48)\\ &\Rightarrow \rho(1) = \frac{\varphi_1}{1-\varphi_2} && (7.49)\\ \rho(2) &= \frac{\varphi_1^2}{1-\varphi_2} + \varphi_2 = \frac{\varphi_1^2 + \varphi_2\,(1-\varphi_2)}{1-\varphi_2} && (7.50)\\ \rho(k) &= \varphi_1\rho(k-1) + \varphi_2\rho(k-2), k \geq 2. && (7.51)\end{aligned}$$

Constraints on $\rho(1)$ and $\rho(2)$ entailed by stationarity:

$$\begin{aligned}|\rho(1)| &< 1, |\rho(2)| < 1 && (7.52)\\ \rho(1)^2 &< \frac{1}{2}[1+\rho(2)]\,; && (7.53)\end{aligned}$$

see Box and Jenkins (1976, p. 61).

7.5 Explicit form for the autocorrelations

The autocorrelations of an $AR(p)$ process satisfy the equation

$$\rho(k) = \sum_{j=1}^{p}\varphi_j\rho(k-j), k \geq 1, \tag{7.54}$$

where $\rho(0) = 1$ and $\rho(-k) = \rho(k)$, or equivalently

$$\varphi(B)\rho(k) = 0\,,\ k \geq 1. \tag{7.55}$$

The autocorrelations can be obtained by solving the homogeneous difference equation (7.54).

The polynome $\varphi(z)$ has m distinct non-zero roots z_1^*, ... , z_m^* (where $1 \leq m \leq p$) with multiplicities p_1, ... , p_m (where $\sum_{j=1}^{m} p_j = p$), so that $\varphi(z)$ can be written

$$\varphi(z) = (1 - G_1 z)^{p_1}(1 - G_2 z)^{p_2}...(1 - G_m z)^{p_m} \tag{7.56}$$

where $G_j = 1/z_j^*$, $j = 1$, ... , m. The roots are real or complex numbers. If z_j^* is a complex (non real) root, its conjugate $\bar{z}_j^*$ is also a root. Consequently, the solutions of equation (7.54) have the general form

$$\rho(k) = \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j - 1} A_{j\ell} k^{\ell} \right) G_j^k, k \geq 1, \tag{7.57}$$

where the $A_{j\ell}$ are (possibly complex) constants which can be determined from the values p autocorrelations. We can easily find $\rho(1)$, ... , $\rho(p)$ from the Yule-Walker equations.

If we write $G_j = r_j e^{i\theta_j}$, where $i = \sqrt{-1}$ while r_j and θ_j are real numbers $(r_j > 0)$,we see that

$$\begin{aligned} \rho(k) &= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j - 1} A_{j\ell}\ k^{\ell} \right) r_j^k e^{i\theta_j k} \\ &= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j - 1} A_{j\ell}\ k^{\ell} \right) r_j^k [\cos(\theta_j k) + i\ \sin(\theta_j k)] \\ &= \sum_{j=1}^{m} \left(\sum_{\ell=0}^{p_j - 1} A_{j\ell}\ k^{\ell} \right) r_j^k \cos(\theta_j k). \end{aligned} \tag{7.58}$$

By stationarity, $0 < |G_j| = r_j < 1$ so that $\rho(k) \to 0$ when $k \to \infty$. The autocorrelations decrease at an exponential rate with oscillations.

7.6 $MA(\infty)$ representation of an $AR(p)$ process

We have seen that a weakly stationary process

$$\varphi(B)\tilde{X}_t = u_t \tag{7.59}$$

where $\varphi(B) = 1 - \varphi_1 B - ... - \varphi_p B^p$, can be written

$$\tilde{X}_t = \psi(B) u_t \tag{7.60}$$

with

$$\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j \tag{7.61}$$

To compute the coefficients ψ_j, it is sufficient to note that

$$\varphi(B)\psi(B) = 1. \tag{7.62}$$

Defining $\psi_j = 0$ for $j < 0$, we see that

$$\begin{aligned}
\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right)\left(\sum_{j=-\infty}^{\infty} \psi_j B^j\right) &= \sum_{j=-\infty}^{\infty} \psi_j \left(B^j - \sum_{k=1}^{p} \varphi_k B^{j+k}\right) \\
&= \sum_{j=-\infty}^{\infty} \left(\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k}\right) B^j \\
&= \sum_{j=-\infty}^{\infty} \tilde{\psi}_j \; B^j = 1.
\end{aligned} \tag{7.63}$$

Thus $\tilde{\psi}_j = 1$, if $j = 0$, and $\tilde{\psi}_j = 0$, if $j \neq 0$. Consequently,

$$\varphi(B)\psi_j = \psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} \quad \begin{aligned} &= 1\,, \text{ if } j = 0 \\ &= 0\,, \text{ if } j \neq 0, \end{aligned} \tag{7.64}$$

where $B^k \psi_j \equiv \psi_{j-k}$. Since $\psi_j = 0$ for $j < 0$, we see that:

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, j \geq 1.
\end{aligned} \tag{7.65}$$

More explicitly,

$$\begin{aligned}
\psi_0 &= 1\,, \\
\psi_1 &= \varphi_1 \psi_0 = \varphi_1\,, \\
\psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 = \varphi_1^2 + \varphi_2\,, \\
\psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3 = \varphi_1^3 + 2\,\varphi_2 \varphi_1 + \varphi_3\,, \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\psi_p &= \sum_{k=1}^{p} \varphi_k \psi_{j-k} \,, \\
\psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, j \geq p+1 \,. \qquad (7.66)
\end{aligned}$$

Under the stationarity condition [roots of $\varphi(z) = 0$ outside the unit circle], the coefficients ψ_j decline at an exponential rate as $j \to \infty$, possibly with oscillations.

Given the representation

$$\tilde{X}_t = \psi(B) u_t = \sum_{j=0}^{\infty} \psi_j u_{t-j} \qquad (7.67)$$

we can easily compute the autocovariances and autocorrelations of X_t :

$$Cov(X_t, X_{t+k}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} \,, \qquad (7.68)$$

$$Corr(X_t, X_{t+k}) = \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|} / \sum_{j=0}^{\infty} \psi_j^2 \,. \qquad (7.69)$$

However, this has the inconvenient of requiring one to compute limits of series.

7.7 Partial autocorrelations

The Yule-Walker equations allow one to determine the autocorrelations from the coefficients $\varphi_1, \ldots, \varphi_p$. In the same way we can determine $\varphi_1, \ldots, \varphi_p$ from the autocorrelations

$$\rho(k) = \sum_{j=1}^{p} \varphi_j \rho(k-j), k = 1, 2, 3, \ldots \qquad (7.70)$$

Taking into account the fact that $\rho(0) = 1$ and $\rho(-k) = \rho(k)$, we find an $AR(p)$ process:

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \ldots & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix} \qquad (7.71)$$

or, in more compact notation,

$$P_p\,\bar{\phi}_p = \bar{\rho}_p. \tag{7.72}$$

It follows that

$$P_k\bar{\phi}_k = \bar{\rho}_k, k = 1, 2, 3, ... \tag{7.73}$$

where $\bar{\phi}_k = (\varphi_{k1}, \varphi_{k2}, ... , \varphi_{kk})'$, so that we can solve for $\bar{\phi}_k$:

$$\bar{\phi}_k = P_k^{-1}\bar{\rho}_k. \tag{7.74}$$

[If $\sigma^2 > 0$, we can show that P_k^{-1} exists, $\forall k \geq 1$]. For an $AR(p)$ process, we see easily

$$\varphi_{kk} = 0, \forall k \geq p + 1. \tag{7.75}$$

The coefficients φ_{kk} are called the lag- k *partial autocorrelations*.

Particular values of φ_{kk} [setting $\rho_k = \rho(k)$] :

$$\varphi_{11} = \rho_1, \tag{7.76}$$

$$\varphi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}, \tag{7.77}$$

$$\varphi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}}. \tag{7.78}$$

7.8 Durbin-Levinson recurrence formula

The partial autocorrelations may be computed using the following recursive formulae:

$$\varphi_{k+1,\,k+1} = \frac{\rho\,(k+1) - \sum_{j=1}^{k}\varphi_{kj}\rho\,(k+1-j)}{1 - \sum_{j=1}^{k}\varphi_{kj}\rho\,(j)}, \tag{7.79}$$

$$\varphi_{k+1,\,j} = \varphi_{kj} - \varphi_{k+1,\,k+1}\varphi_{k,\,k-j+1}, j = 1, 2, ..., k. \tag{7.80}$$

Given $\rho(1), ... , \rho(k+1)$ and $\varphi_{k1}, ... , \varphi_{kk}$, we can compute $\varphi_{k+1,\,j},\; j = 1, ... , k+1$. See Durbin (1960) and Box and Jenkins (1976, pp. 82-84).

8. Mixed processes

Consider a process $\{X_t : t \in \mathbb{Z}\}$ which satisfies the equation:

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j \, X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{8.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. Using operational notation,

$$\varphi(B) X_t = \bar{\mu} + \theta(B) u_t. \tag{8.2}$$

8.1 Stationarity conditions

If the polynome $\varphi(z) = 1 - \varphi_1 z - ... - \varphi_p z^p$ has all its roots outside the unit circle, the equation (8.1) has one and only one weakly stationary solution, which can be written:

$$X_t = \mu + \frac{\theta(B)}{\varphi(B)} u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \, , \tag{8.3}$$

where

$$\begin{aligned} \mu &= \bar{\mu}/\varphi(B) = \bar{\mu}/(1 - \sum_{j=1}^{p} \varphi_j) \, , & (8.4) \\ \frac{\theta(B)}{\varphi(B)} &\equiv \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j \, . & (8.5) \end{aligned}$$

The coefficients ψ_j are obtained by solving the equation

$$\varphi(B) \psi(B) = \theta(B). \tag{8.6}$$

In this case, we also have:

$$E(X_{t-j} u_t) = 0, \forall j \geq 1. \tag{8.7}$$

The ψ_j coefficients may be computed in the following way (setting $\theta_0 = -1$) :

$$\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right) \left(\sum_{j=0}^{\infty} \psi_j B^j\right) = 1 - \sum_{j=1}^{q} \theta_j B^j = -\sum_{j=1}^{q} \theta_j B^j \tag{8.8}$$

hence

$$\begin{aligned} \varphi(B) \psi_j &= -\theta_j \, , \; j = 0, 1, ..., q \\ &= 0 \, , j \geq q + 1, \end{aligned} \tag{8.9}$$

where $\psi_j = 0$, for $j < 0$. Consequently,

$$\begin{aligned}\psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k} - \theta_j, \quad j = 0, 1, ..., q \\ &= \sum_{k=1}^{p} \varphi_k \psi_{j-k} \,, \qquad j \geq q+1,\end{aligned} \tag{8.10}$$

and

$$\begin{aligned}\psi_0 &= 1\,, \\ \psi_1 &= \varphi_1 \psi_0 - \theta_1 = \varphi_1 - \theta_1\,, \\ \psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 - \theta_2 = \varphi_1 \psi_1 + \varphi_2 - \theta_2 = \varphi_1^2 - \varphi_1 \theta_1 + \varphi_2 - \theta_2\,, \\ &\vdots \\ \psi_j &= \sum_{k=1}^{p} \varphi_k \psi_{j-k}, j \geq q+1\,.\end{aligned} \tag{8.11}$$

The ψ_j coefficients behave like the autocorrelations of an $AR(p)$ process, except for the initial coefficients ψ_1, ... , ψ_q.

8.2 Autocovariances and autocorrelations

Suppose:

a) the process X_t is second-order stationary with $\sum_{j=1}^{p} \varphi_j \neq 1$;

b) $E(X_{t-j}u_t) = 0$, $\forall j \geq 1$. (8.12)

By the stationarity assumption,

$$E(X_t) = \mu, \forall t, \tag{8.13}$$

hence

$$\mu = \bar{\mu} \ + \sum_{j=1}^{p} \varphi_j \mu \tag{8.14}$$

and

$$E(X_t) = \mu = \bar{\mu} / \left(1 - \sum_{j=1}^{p} \varphi_j \right) . \tag{8.15}$$

The mean is the same as in the case of a pure $AR(p)$ process. The $MA(q)$ part has no effect on the mean. Let us now rewrite the process in the form

$$\tilde{X}_t = \sum_{j=1}^{p} \varphi_j \tilde{X}_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{8.16}$$

where $\tilde{X}_t = X_t - \mu$. Consequently,

$$\begin{aligned}
\tilde{X}_{t+k} &= \sum_{j=1}^{p} \varphi_j \ \tilde{X}_{t+k-j} + u_{t+k} - \sum_{j=1}^{q} \theta_j u_{t+k-j} \,, & (8.17)\\
E(\tilde{X}_t \ \tilde{X}_{t+k}) &= \sum_{j=1}^{p} \varphi_j E(\tilde{X}_t \ \tilde{X}_{t+k-j}) + E(\tilde{X}_t \ u_{t+k}) - \sum_{j=1}^{q} \theta_j E(\tilde{X}_t \ u_{t+k-j}) \,, & (8.18)\\
\gamma(k) &= \sum_{j=1}^{p} \varphi_j \gamma(k-j) + \gamma_{xu}(k) - \sum_{j=1}^{q} \theta_j \gamma_{xu}(k-j) \,, & (8.19)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{xu}(k) &= E(\tilde{X}_t \ u_{t+k}) = 0 \,, \quad \text{if } k \geq 1 \,, \\
&\neq 0 \,, \qquad\qquad\qquad\ \text{if } k \leq 0 \,, \\
\gamma_{xu}(0) &= E(\tilde{X}_t \ u_t) = \sigma^2.
\end{aligned} \tag{8.20}$$

For $k \geq q+1$,

$$\begin{aligned}
\gamma(k) &= \sum_{j=1}^{p} \varphi_j \gamma(k-j), & (8.21)\\
\rho(k) &= \sum_{j=1}^{p} \varphi_j \rho(k-j). & (8.22)
\end{aligned}$$

The variance is given by

$$\gamma(0) = \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j) \tag{8.23}$$

hence

$$\gamma(0) = \left[\sigma^2 \ - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)\right] \Big/ \left[1 - \sum_{j=1}^{p} \varphi_j \rho(j)\right]. \tag{8.24}$$

In operational notation, the autocovariances satisfy the equation

$$\varphi(B)\gamma(k) = \theta(B)\gamma_{xu}(k) \; , k \geq 0, \tag{8.25}$$

where $\gamma(-k) = \gamma(k)$, $B^j\gamma(k) \equiv \gamma(k-j)$ and $B^j\gamma_{xu}(k) \equiv \gamma_{xu}(k-j)$. In particular,

$$\begin{aligned} \varphi(B)\gamma(k) &= 0 \; , k \geq q+1, & (8.26) \\ \varphi(B)\rho(k) &= 0 \; , k \geq q+1. & (8.27) \end{aligned}$$

To compute the autocovariances, we can solve the equations (8.19) for $k = 0$, 1, ... , p, and then apply (8.21). The autocorrelations of an process ARMA(p, q) process behave like those of an $AR(p)$ process, except that initial values are modified.

8.3 Example ARMA(1, 1) process

$$X_t = \bar{\mu} + \varphi_1 X_{t-1} + u_t - \theta_1 u_{t-1} \; , |\varphi_1| < 1 \tag{8.28}$$

$$\tilde{X}_t - \varphi_1 \; \tilde{X}_{t-1} = u_t - \theta_1 u_{t-1} \tag{8.29}$$

where $\tilde{X}_t = X_t - \mu$. We have

$$\begin{aligned} \gamma(0) &= \varphi_1\gamma(1) + \gamma_{xu}(0) - \theta_1\gamma_{xu}(-1), & (8.30) \\ \gamma(1) &= \varphi_1\gamma(0) + \gamma_{xu}(1) - \theta_1\gamma_{xu}(0) & (8.31) \end{aligned}$$

and

$$\begin{aligned} \gamma_{xu}(1) &= 0, & (8.32) \\ \gamma_{xu}(0) &= \sigma^2, & (8.33) \\ \gamma_{xu}(-1) &= E(\tilde{X}_t u_{t-1}) = \varphi_1 E(\tilde{X}_{t-1}u_{t-1}) + E(u_t u_{t-1}) - \theta_1 E(u_{t-1}^2) & \\ &= \varphi_1\gamma_{xu}(0) - \theta_1\sigma^2 = (\varphi_1 - \theta_1)\sigma^2 & (8.34) \end{aligned}$$

Thus,

$$\begin{aligned} \gamma(0) &= \varphi_1\gamma(1) + \sigma^2 - \theta_1(\varphi_1 - \theta_1)\sigma^2 & \\ &= \varphi_1\gamma(1) + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2, & (8.35) \end{aligned}$$

$$\begin{aligned} \gamma(1) &= \varphi_1\gamma(0) - \theta_1\sigma^2 & \\ &= \varphi_1\{\varphi_1\gamma(1) + [1 - \theta_1(\varphi_1 - \theta_1)]\sigma^2\} - \theta_1\sigma^2 \; , & (8.36) \end{aligned}$$

hence

$$\begin{aligned}
\gamma(1) &= \{\varphi_1[1-\theta_1(\varphi_1-\theta_1)]-\theta_1\}\sigma^2/(1-\varphi_1^2) \\
&= \{\varphi_1-\theta_1\varphi_1^2+\varphi_1\theta_1^2-\theta_1\}\sigma^2/(1-\varphi_1^2) \\
&= (1-\theta_1\varphi_1)(\varphi_1-\theta_1)\sigma^2/(1-\varphi_1^2)\,. \qquad (8.37)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\gamma(0) &= \varphi_1\gamma(1)+[1-\theta_1(\varphi_1-\theta_1)]\sigma^2 \\
&= \varphi_1\frac{(1-\theta_1\varphi_1)\,(\varphi_1-\theta_1)\,\sigma^2}{1-\varphi_1^2}+[1-\theta_1(\varphi_1-\theta_1)]\sigma^2 \\
&= \frac{\sigma^2}{1-\varphi_1^2}\,\{\varphi_1(1-\theta_1\varphi_1)(\varphi_1-\theta_1)+(1-\varphi_1^2)[1-\theta_1(\varphi_1-\theta_1)]\} \\
&= \frac{\sigma^2}{1-\varphi_1^2}\,\{\varphi_1^2-\theta_1\varphi_1^3+\varphi_1^2\theta_1^2-\varphi_1\theta_1+1-\varphi_1^2-\theta_1\varphi_1+\theta_1\varphi_1^3+\theta_1^2-\varphi_1^2\,\theta_1^2\} \\
&= \frac{\sigma^2}{1-\varphi_1^2}\,\{1-2\,\varphi_1\theta_1+\theta_1^2\}\,. \qquad (8.38)
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma(0) &= (1-2\,\varphi_1\theta_1+\theta_1^2)\sigma^2/(1-\varphi_1^2)\,, \qquad (8.39) \\
\gamma(1) &= (1-\theta_1\varphi_1)(\varphi_1-\theta_1)\sigma^2/(1-\varphi_1^2)\,, \qquad (8.40) \\
\gamma(k) &= \varphi_1\gamma(k-1),\ \text{for } k\geq 2\,. \qquad (8.41)
\end{aligned}$$

9. Invertibility

9.1 Any second-order stationary $AR(p)$ process can be written under an $MA(\infty)$ form. Similarly, any second-order stationary $ARMA(p,\ q)$ process can also be written under an $MA(\infty)$ form. By analogy, it is natural to ask the question: can a $MA(q)$ or $ARMA(p,\ q)$ process be represented in a purely autoregressive form?

9.2 Consider the process MA(1) :

$$X_t = u_t - \theta_1 u_{t-1}, t\in\mathbb{Z}\,, \qquad (9.1)$$

where $\{u_t : t\in\mathbb{Z}\}\ \sim\ BB(0,\sigma^2)$ and $\sigma^2>0$. We see easily that

$$u_t \;=\; X_t+\theta_1 u_{t-1}$$

$$
\begin{aligned}
&= X_t + \theta_1(X_{t-1} + \theta_1 u_{t-2}) \\
&= X_t + \theta_1 X_{t-1} + \theta_1^2 u_{t-2} \\
&= \sum_{j=0}^{n} \theta_1^j X_{t-j} + \theta_1^{n+1} u_{t-n-1} \qquad (9.2)
\end{aligned}
$$

and

$$
E\left[\left(\sum_{j=0}^{n} \theta_1^j X_{t-j} - u_t\right)^2\right] = E\left[\left(\theta_1^{n+1} u_{t-n-1}\right)^2\right] = \theta_1^{2(n+1)}\sigma^2 \underset{n\to\infty}{\to} 0, \qquad (9.3)
$$

provided $|\theta_1| < 1$. Consequently, the series $\sum_{j=0}^{n} \theta_1^j X_{t-j}$ converges in $q.m.$ to u_t if $|\theta_1| < 1$. In other words, when $|\theta_1| < 1$, we can write

$$
\sum_{j=0}^{\infty} \theta_1^j X_{t-j} = u_t, t \in \mathbb{Z} \,, \qquad (9.4)
$$

or

$$
(1 - \theta_1 B)^{-1} X_t = u_t, t \in \mathbb{Z} \,, \qquad (9.5)
$$

where $(1 - \theta_1 B)^{-1} = \sum_{j=0}^{\infty} \theta_1^j B^j$. The condition $|\theta_1| < 1$ is equivalent to having the roots of the equation $1 - \theta_1 z = 0$ outside the unit circle. If $\theta_1 = 1$,

$$
X_t = u_t - u_{t-1} \qquad (9.6)
$$

and the series

$$
(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} \theta_1^j X_{t-j} = \sum_{j=0}^{\infty} X_{t-j} \qquad (9.7)
$$

does not converge, for $E(X_{t-j}^2)$ does not converge to 0 as $j \to \infty$. Similarly, if $\theta_1 = -1$,

$$
X_t = u_t + u_{t-1} \qquad (9.8)
$$

and the series

$$
(1 - \theta_1 B)^{-1} X_t = \sum_{j=0}^{\infty} (-1)^j X_{t-j} \qquad (9.9)
$$

does not converge either. These models are not invertible.

9.3 Theorem *(Invertibility condition for a MA process) : Let $\{X_t : t \in \mathbb{Z})$ be a second-*

order stationary process such that

$$X_t = \mu + \theta(B)u_t \tag{9.10}$$

where $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$. *Then the process* X_t *satisfies an equation of the form*

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\mu} + u_t \tag{9.11}$$

iff the roots of the polynome $\theta(z)$ *are outside the unit circle. Further, when the representation (*9.11*) exists, we have:*

$$\bar{\phi}(B) = \theta(B)^{-1},\ \bar{\mu} = \theta(B)^{-1}\mu = \mu / \left(1 - \sum_{j=1}^{q} \theta_j\right). \tag{9.12}$$

9.4 Corollary *(Invertibility of an* $ARMA$ *process) : Let* $\{X_t : t \in \mathbb{Z}\}$ *be a second-order stationary* $ARMA$ *process that satisfies the equation*

$$\varphi(B)X_t = \bar{\mu} + \theta(B)u_t \tag{9.13}$$

where $\varphi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p$ *and* $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$. *Then the process* X_t *satisfies an equation of the form*

$$\sum_{j=0}^{\infty} \bar{\phi}_j X_{t-j} = \bar{\bar{\mu}} + u_t \tag{9.14}$$

iff the roots du polynome $\theta(z)$ *are outside the unit circle. Further, when the representation (*9.14*) exists, we have:*

$$\bar{\phi}(B) = \theta(B)^{-1}\varphi(B), \bar{\bar{\mu}} = \theta(B)^{-1}\bar{\mu} = \mu / \left(1 - \sum_{j=1}^{q} \theta_j\right). \tag{9.15}$$

10. Wold representation

10.1 We have seen that all second-order ARMA processes can be written in a causal $MA(\infty)$ form. This property indeed holds for all second-order stationary processes.

10.2 Theorem *(Wold) : Let $\{X_t, \, t \in \mathbb{Z}\}$ be a second-order stationary process such that $E(X_t) = \mu$. Then X_t can be written in the form*

$$X_t = \mu \; + \sum_{j=0}^{\infty} \psi_j u_{t-j} + v_t \tag{10.1}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, $E(u_t X_{t-j}) = 0, \forall j \geq 1$, and $\{v_t : t \in \mathbb{Z}\}$ is a process deterministic such that $E(v_t) = 0$ and $E(u_s v_t) = 0, \; \forall s, \, t$. Further, if $\sigma^2 > 0$, the sequences $\{\psi_j\}$ and $\{u_t\}$ are unique, and

$$u_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t-1}, \tilde{X}_{t-2}, ...) \tag{10.2}$$

where $\tilde{X}_t = X_t - \mu$.

PROOF: See Anderson (1971, Section 7.6.3, pp. 420-421).

10.3 If $E(u_t^2) > 0$ in Wold representation, we say the process X_t is *regular*. v_t is called the *deterministic component of* the process while $\sum_{j=0}^{\infty} \psi_j u_{t-j}$ is its *indeterministic component*. When $v_t = 0, \forall t$, the process X_t is said to be *strictly indeterministic*.

10.4 Corollary *(Forward Wold representation) : Let $\{X_t \, : \, t \, \in \, \mathbb{Z}\}$ be second-order a stationary process such that $E(X_t) = \mu$. Then X_t can be written in the form*

$$X_t = \mu \; + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j} + \bar{v}_t \tag{10.3}$$

where $\{\bar{u}_t : t \in \mathbb{Z}\} \sim BB(0, \bar{\sigma}^2)$, $\sum_{j=0}^{\infty} \bar{\psi}_j^2 < \infty$, $E(\bar{u}_t X_{t+j}) = 0$, $\forall j \geq 1$, and $\{\bar{v}_t : t \in \mathbb{Z}\}$ is a deterministic (with respect to $\bar{v}_{t+1}, \, \bar{v}_{t+2}$, ...) such that $E(\bar{v}_t) = 0$ and $E(\bar{u}_s \bar{v}_t) = 0$, $\forall s, t$. Further, if $\bar{\sigma}^2 > 0$, the sequences $\{\bar{\psi}_j\}$ and $\{\bar{u}_t\}$ are uniquely defined, and

$$\bar{u}_t = \tilde{X}_t - P(\tilde{X}_t | \tilde{X}_{t+1}, \tilde{X}_{t+2}, ...) \tag{10.4}$$

where $\tilde{X}_t = X_t - \mu$.

PROOF. The result follows on applying Wold theorem to the process $Y_t \equiv X_{-t}$ qui is also second-order stationary. $Q.E.D.$ □

11. Generating functions and spectral density

11.1 Generating functions constitute a convenient technique representing or finding the autocovariance structure of a stationary process.

11.2 Definition *(Generating function) : Let* $(a_k : k = 0,\ 1,\ 2,\ ...)$ *and* $(b_k : k = ...,-1,\ 0,\ 1,\ ...)$ *two sequences of complex numbers. Let* $D(a) \subseteq \mathbf{C}$ *the set of points* $z \in \mathbf{C}$ *for which the series* $\sum_{k=0}^{\infty} a_k z^k$ *converges, and let* $D(b) \subseteq \mathbf{C}$ *the set of points* z *for which where the series* $\sum_{k=-\infty}^{\infty} b_k z^k$ *converges. Then the functions*

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, z \in D(a) \tag{11.1}$$

and

$$b(z) = \sum_{k=-\infty}^{\infty} b_k z^k, z \in D(b) \tag{11.2}$$

are called the generating functions of the sequences a_k *and* b_k *respectively.*

11.3 Proposition *(Convergence annulus of a generating function) : Let* $(a_k : k \in \mathbb{Z})$ *be a sequence of complex numbers. Then the generating function*

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k \tag{11.3}$$

converges for $R_1 < |z| < R_2$ *where*

$$R_1 = \limsup_{k\to\infty} |a_{-k}|^{1/k}, \tag{11.4}$$

$$R_2 = 1/\left[\limsup_{k\to\infty}|a_k|^{1/k}\right], \tag{11.5}$$

and diverges for $|z| < R_1$ *or* $|z| > R_2$*. If* $R_2 < R_1$*,* $a(z)$ *converges nowhere and, if* $R_1 = R_2$*,* $a(z)$ *diverges everywhere except possibly, for* $|z| = R_1 = R_2$*. Further, when* $R_1 < R_2$*, the coefficients* a_k *are uniquely defined, and*

$$a_k = \frac{1}{2\pi i}\int_C \frac{a(z)\ dz}{(z - z_0)^{k+1}}, \ k = 0, \pm1, \pm2, ... \tag{11.6}$$

where $C = \{z \in \mathbf{C} : |z - z_0| = R\}$ *and* $R_1 < R < R_2$ *.*

11.4 Proposition *(Sums and products of generating functions) : Let* $(a_k : k \in \mathbb{Z})$ *and* $(b_k \in \mathbb{Z})$ *two sequences of complex numbers such that the generating functions* $a(z)$ *and* $b(z)$ *converge for* $R_1 < |z| < R_2$*, where* $0 \le R_1 < R_2 \le \infty$*. Then,*

(1) *the generating function of the sum* $c_k = a_k + b_k$ *is* $c(z) = a(z) + b(z)$*;*

(2) *if the product sequence*

$$d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} \tag{11.7}$$

converges for any k*, the generating function of the sequence* d_k *is*

$$d(z) = a(z)b(z). \tag{11.8}$$

Further, the series $c(z)$ *and* $d(z)$ *converge for* $R_1 < |z| < R_2$*.*

11.5 We will be especially interested by generating functions of autocovariances γ_k and autocorrelations ρ_k of a second-order stationary process X_t:

$$\gamma_x(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \tag{11.9}$$

$$\rho_x(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \gamma_x(z)/\gamma_0. \tag{11.10}$$

We see immediately that the generating function with a white noise $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$ is constant::

$$\gamma_u(z) = \sigma^2, \rho_u(z) = 1. \tag{11.11}$$

11.6 Proposition *(Convergence of the generating function of the autocovariances): Let $\gamma_k, k \in \mathbb{Z}$, the autocovariances of a second-order stationary process X_t, and ρ_k, $k \in \mathbb{Z}$, the corresponding autocorrelations.*

(1) *If $R \equiv \limsup_{k\to\infty} |\rho_k|^{1/k} < 1$, the generating functions $\gamma_x(z)$ and $\rho_x(z)$ converge for $R < |z| < 1/R$.*

(2) *If $R = 1$, the functions $\gamma_x(z)$ and $\rho_x(z)$ diverge everywhere, except possibly on the circle $|z| = 1$.*

(3) *If $\sum_{k=0}^{\infty} |\rho_k| < \infty$, the functions $\gamma_x(z)$ and $\rho_x(z)$ converge absolutely and uniformly on the circle $|z| = 1$.*

11.7 Proposition *(Unicity) : Let γ_k and ρ_k, $k \in \mathbb{Z}$, autocovariance and autocorrelation sequences such that*

$$\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k = \sum_{k=-\infty}^{\infty} \gamma'_k z^k, \tag{11.12}$$

$$\rho(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \sum_{k=-\infty}^{\infty} \rho'_k z^k \tag{11.13}$$

where the series considered converge for $R < |z| < 1/R$, where $R \geq 0$. Then $\gamma_k = \gamma'_k$ and $\rho_k = \rho'_k$ for any $k \in \mathbb{Z}$.

11.8 Proposition (*Generating function of the autocovariances of a $MA(\infty)$ process*) : *Let $\{X_t : t \in \mathbb{Z}\}$ a second-order stationary process such that*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \tag{11.14}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. *If the series*

$$\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \tag{11.15}$$

and $\psi(z^{-1})$ *converge absolutely, then*

$$\gamma_x(z) = \sigma^2 \psi(z)\psi(z^{-1}). \tag{11.16}$$

11.9 Corollary (*Generating function of the autocovariances of an* $ARMA$ *process*) *: Let* $\{X_t : t \in \mathbb{Z}\}$ *a second-order stationary and causal* $ARMA(p, q)$ *process, such that*

$$\varphi(B)X_t = \bar{\mu} + \theta(B)u_t \tag{11.17}$$

where $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, $\varphi(z) = 1 - \varphi_1 z - ... - \varphi_p z^p$ *and* $\theta(z) = 1 - \theta_1 z - ... - \theta_q z^q$. *Then the generating function of the autocovariances of* X_t *is*

$$\gamma_x(z) = \sigma^2 \frac{\theta\,(z)\;\;\theta\,(z^{-1})}{\varphi\,(z)\;\;\varphi\,(z^{-1})} \tag{11.18}$$

for $R < |z| < 1/R$, *where*

$$0 < R = \max\{|G_1|, |G_2|, ..., |G_p|\} < 1 \tag{11.19}$$

and $G_1^{-1}, G_2^{-1}, ..., G_p^{-1}$ *are the roots of the polynome* $\varphi(z)$.

11.10 Proposition *(Generating function of the autocovariances of a filtered process) : Let* $\{X_t : t \in \mathbb{Z}\}$ *a second-order stationary process and*

$$Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}, t \in \mathbb{Z}, \tag{11.20}$$

where $(c_j : j \in \mathbb{Z})$ *is a sequence of real constants such that* $\sum\limits_{j=-\infty}^{\infty} |c_j| < \infty$. *If the series* $\gamma_x(z)$ *and* $c(z) = \sum\limits_{j=-\infty}^{\infty} c_j z^j$ *converge absolutely, then*

$$\gamma_y(z) = c(z)c(z^{-1})\gamma_x(z). \tag{11.21}$$

11.11 Definition *(Spectral density) : Let X_t a second-order stationary process such that the generating function of the autocovariances $\gamma_x(z)$ converge for $|z| = 1$. The spectral density of the process X_t is the function*

$$\begin{aligned} f_x(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)\right] \\ &= \frac{\gamma_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\gamma_k \cos(\omega k) \end{aligned} \tag{11.22}$$

where the coefficients γ_k are the autocovariances of the process X_t. The function $f_x(\omega)$ is defined for all the values of ω such that the series $\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)$ converges.

11.12 Remark If the series $\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)$ converges, it is immediate that $\gamma_x(e^{-i\omega})$ converge and

$$f_x(\omega) = \frac{1}{2\pi}\gamma_x(e^{-i\omega}) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k e^{-i\omega k} \tag{11.23}$$

where $i = \sqrt{-1}$.

11.13 Proposition (*Convergence and properties of the spectral density*) *: Let γ_k, $k \in \mathbb{Z}$, be an autocovariance function such that $\sum_{k=0}^{\infty}|\gamma_k| < \infty$. Then*

(1) *the series*

$$f_x(\omega) = \frac{\gamma_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\gamma_k \cos(\omega k) \tag{11.24}$$

converges absolutely and uniformly in ω ;

(2) *the function $f_x(\omega)$ is continuous ;*

(3) $f_x(\omega + 2\pi) = f_x(\omega)$ *and* $f_x(-\omega) = f_x(\omega), \forall\omega$ *;*

(4) $\gamma_k = \int_{-\pi}^{\pi} f_x(\omega)\cos(\omega k)d\omega, \forall k$ *;*

(5) $f_x(\omega) \geq 0$ *;*

(6) $\gamma_0 = \int_{-\pi}^{\pi} f_x(\omega)d\omega$.

11.14 Proposition *(Spectral densities of special processes) : Let* $\{X_t : t \in \mathbb{Z}\}$ *be a second-order stationary process with autocovariances* γ_k, $k \in \mathbb{Z}$.

(1) *If* $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ *where* $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$ *and* $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, *then*

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(e^{i\omega})\psi(e^{-i\omega}) = \frac{\sigma^2}{2\pi} |\psi(e^{i\omega})|^2. \tag{11.25}$$

(2) *If* $\varphi(B)X_t = \bar{\mu} + \theta(B)u_t$ *,where* $\varphi(B) = 1 - \varphi_1 B - ... - \varphi_p B^p,\ \theta(B) = 1 - \theta_1 B - ... - \theta_q B^q$ *and* $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, *then*

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta\left(e^{i\omega}\right)}{\varphi\left(e^{i\omega}\right)} \right|^2 \tag{11.26}$$

(3) *If* $Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}$ *where* $(c_j : j \in \mathbb{Z})$ *is a sequence of real constants such that* $\sum_{j=-\infty}^{\infty} |c_j| < \infty$ *, and if* $\sum_{k=0}^{\infty} |\gamma_k| < \infty$ *, then*

$$f_y(\omega) = |c(e^{i\omega})|^2 f_x(\omega). \tag{11.27}$$

12. Inverse autocorrelations

12.1 Definition *(Autocorrelations inverses) : Let* $f_x(\omega)$ *the spectral density of a second-order stationary process* $\{X_t : t \in \mathbb{Z}\}$. *If the function* $1/f_x(\omega)$ *is also a spectral density, the autocovariances* $\gamma_x^{(I)}(k)$, $k \in \mathbb{Z}$, *associated with the inverse spectrum inverse* $1/f_x(\omega)$ *are called the* inverse autocovariances *of the process* X_t, $i.e.$

$$\gamma_x^{(I)}(k) = \int_{-\pi}^{\pi} \frac{1}{f_x\left(\omega\right)} \cos(\omega k)d\omega, k \in \mathbb{Z}. \tag{12.1}$$

12.2 The inverse autocovariances satisfy the equation

$$\frac{1}{f_x(\omega)} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_x^{(I)}(k)\cos(\omega k) = \frac{1}{2\pi}\,\gamma_x^{(I)}(0) + \frac{1}{\pi}\sum_{k=1}^{\infty}\gamma_x^{(I)}\cos(\omega k). \tag{12.2}$$

The inverse autocorrelations are

$$\rho_x^{(I)}(k) = \gamma_x^{(I)}(k)/\gamma_x^{(I)}(0), k \in \mathbb{Z}. \tag{12.3}$$

12.3 A sufficient condition for the function $1/f_x(\omega)$ to be a spectral density is that the function $1/f_x(\omega)$ be continuous on the interval $-\pi \leq \omega \leq \pi$, which entails that $f_x(\omega) > 0$, $\forall\omega$.

12.4 If the process X_t is a second-order stationary $ARMA(p, q)$ process such that

$$\varphi_p(B)X_t = \bar{\mu} + \theta_q(B)u_t \tag{12.4}$$

where $\varphi_p(B) = 1-\varphi_1B - ... -\varphi_pB^p$ and $\theta_q(B) = 1-\theta_1B - ... -\theta_qB^q$ are des polynomes which have all their roots outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, then

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\left|\frac{\theta_q(e^{i\omega})}{\varphi_p(e^{i\omega})}\right|^2 \tag{12.5}$$

and

$$\frac{1}{f_x(\omega)} = \frac{2\pi}{\sigma^2}\left|\frac{\varphi_p(e^{i\omega})}{\theta_q(e^{i\omega})}\right|^2 . \tag{12.6}$$

The inverse autocovariances $\gamma_x^{(I)}(k)$ are the autocovariances associated with the model

$$\theta_q(B)X_t = \bar{\bar{\mu}} + \varphi_p(B)v_t \tag{12.7}$$

where $\{v_t : t \in \mathbb{Z}\} \sim BB(0 , 1/\sigma^2)$ and $\bar{\bar{\mu}}$ is some constant. Consequently, the inverse autocorrelations of an $ARMA(p, q)$ process behave like the autocorrelations of an $ARMA(q, p)$. For an process $AR(p)$ process,

$$\rho_x^{(I)}(k) = 0\,, \text{ for } k > p. \tag{12.8}$$

For a $MA(q)$ process, the inverse partial autocorrelations ($i.e.$ the partial autocorrelations associated with the inverse autocorrelations) are equal to zero for $k > q$. These properties can be used for identifying the order of a process.

13. Multiplicity of representations

13.1. Backward representation ARMA models

By the backward Wold theorem, we know that any strictly indeterministic second-order stationary process $X_t : t \in \mathbb{Z}\}$ can be written in the form

$$X_t = \mu \ + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j} \tag{13.1}$$

where $\bar{u}_t$ is a white noise such that $E(X_{t-j}\bar{u}_t) = 0$, $\forall j \geq 1$. In particular, if

$$\varphi_p(B)(X_t - \mu) = \theta_q(B)u_t \tag{13.2}$$

where the polynomes $\varphi_p(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p$ and $\theta_q(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$ have all their roots outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$, the spectral density of X_t is

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q\left(e^{i\omega}\right)}{\varphi_p\left(e^{i\omega}\right)} \right|^2 . \tag{13.3}$$

Consider the process

$$Y_t = \frac{\varphi_p\left(B^{-1}\right)}{\theta_q\left(B^{-1}\right)} \left(X_t - \mu\right) = \sum_{j=0}^{\infty} c_j (X_{t+j} - \mu). \tag{13.4}$$

Pour the Proposition **11.14**, the spectral density of Y_t is

$$f_y(\omega) = \left| \frac{\varphi_p\left(e^{i\omega}\right)}{\theta_q\left(e^{i\omega}\right)} \right|^2 f_x(\omega) = \frac{\sigma^2}{2\pi} \tag{13.5}$$

and thus $\{Y_t : t \in \mathbb{Z}\} \sim BB(0, \sigma^2)$. If we define $\bar{u}_t = Y_t$, we see that

$$\frac{\varphi_p\left(B^{-1}\right)}{\theta_q\left(B^{-1}\right)} \left(X_t - \mu\right) = \bar{u}_t \tag{13.6}$$

or

$$\varphi_p(B^{-1})X_t = \bar{\mu} + \theta_q(B^{-1})\bar{u}_t, \tag{13.7}$$

and

$$(10.1.7)X_t - \varphi_1 X_{t+1} - ... - \varphi_p X_{t+p} = \bar{\mu} + \bar{u}_t - \theta_1 \bar{u}_{t+1} - ... - \theta_q \bar{u}_{t+q} \tag{13.8}$$

where $(1 - \varphi_1 - ... - \varphi_p)\mu = \bar{\mu}$. We call (13.6) or (13.8) the backward representation of the X_t process.

13.2. Multiple moving-average representations

Let $\{X_t\} \sim$ ARIMA(p, d, q) . Then

$$W_t = (1 - B)^d X_t \ \sim ARMA(p, q). \tag{13.9}$$

If we suppose that $E(W_t) = 0$, W_t satisfies an equation of the form

$$\varphi_p(B)W_t = \theta_q(B)u_t \tag{13.10}$$

or

$$W_t = \frac{\theta_q(B)}{\varphi_p(B)} u_t = \psi(B)u_t. \tag{13.11}$$

To determine an appropriate $ARMA$ model, one typically estimates the autocorrelations ρ_k. The latter are uniquely determined by the generating function of the autocovariances:

$$\gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2 \frac{\theta_q(z)}{\varphi_p(z)} \frac{\theta_q(z^{-1})}{\varphi_p(z^{-1})}. \tag{13.12}$$

If

$$\theta_q(z) = 1 - \theta_1 z - ... - \theta_q z^q = (1 - H_1 z)...(1 - H_q z) = \prod_{j=1}^{q}(1 - H_j z), \tag{13.13}$$

then

$$\gamma_x(z) = \frac{\sigma^2}{\varphi_p(z)\varphi_p(z^{-1})} \prod_{j=1}^{q}(1 - H_j z)(1 - H_j z^{-1}). \tag{13.14}$$

However

$$\begin{aligned}(1 - H_j z)(1 - H_j z^{-1}) &= 1 - H_j z - H_j z^{-1} + H_j^2 = H_j^2(1 - H_j^{-1}z - H_j^{-1}z^{-1} + H_j^{-2}) \\ &= H_j^2(1 - H_j^{-1}z)(1 - H_j^{-1}z^{-1})\end{aligned} \tag{13.15}$$

hence

$$\begin{aligned}
\gamma_x(z) &= \frac{\left[\sigma^2 \prod_{j=1}^{q} H_j^2\right]}{\varphi_p(z)\,\varphi_p(z^{-1})} \prod_{j=1}^{q} \left(1 - H_j^{-1} z\right)\left(1 - H_j^{-1} z^{-1}\right) \\
&= \bar{\sigma}^2 \frac{\theta_q^{'}(z)\,\theta_q^{'}(z^{-1})}{\varphi_p(z)\,\varphi_p(z^{-1})} \qquad (13.16)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\sigma}^2 &= \sigma^2 \prod_{j=1}^{q} H_j^2, \qquad (13.17) \\
\theta_q'(z) &= \prod_{j=1}^{q} (1 - H_j^{-1} z). \qquad (13.18)
\end{aligned}$$

$\gamma_x(z)$ in (13.16) can be viewed as the generating function of a process of the form

$$\varphi_p(B)W_t = \theta_q'(B)\bar{u}_t = [\prod_{j=1}^{q}(1 - H_j^{-1}B)]\bar{u}_t \qquad (13.19)$$

while $\gamma_x(z)$ in (13.14) is the generating function of

$$\varphi_p(B)W_t = \theta_q(B)u_t = [\prod_{j=1}^{q}(1 - H_j B)]u_t. \qquad (13.20)$$

The processes (13.19) and (13.20) have the same autocovariance function and thus cannot be distinguished by looking at their seconds moments.

13.1 Example

$$(1 - 0.5B)W_t = (1 - 0.2B)(1 + 0.1B)u_t \qquad (13.21)$$

$$(1 - 0.5B)W_t = (1 - 5B)(1 + 10B)\bar{u}_t \qquad (13.22)$$

have the same autocorrelation function.

In general, the models

$$\varphi_p(B)W_t = \left[\prod_{j=1}^{q}(1 - H_j^{\pm 1}B)\right]\bar{u}_t \qquad (13.23)$$

all have the same autocovariance function (and are thus indistinguishable). Since it is easier with an invertible model, we select

$$H_j^* = \begin{cases} H_j \text{ , if } \\ H_j^{-1} \text{ , if } \end{cases} \left| \begin{matrix} H_j \\ H_j \end{matrix} \right| \begin{matrix} < 1 \\ > 1 \end{matrix} , \tag{13.24}$$

where $|H_j| \leq 1$, in order to have an invertible model.

13.3. Redundant parameters

Suppose $\varphi_p(B)$ and $\theta_q(B)$ have a common factor, say $G(B)$:

$$\varphi_p(B) = G(B)\varphi_{p_1}(B), \theta_q(B) = G(B)\theta_{q_1}(B). \tag{13.25}$$

Consider the models

$$\begin{aligned} \varphi_p(B)W_t &= \theta_q(B)u_t & (13.26) \\ \varphi_{p_1}(B)W_t &= \theta_{q_1}(B)u_t. & (13.27) \end{aligned}$$

The $MA(\infty)$ representations of these two models are

$$W_t = \psi(B)u_t, \tag{13.28}$$

where

$$\psi(B) = \frac{\theta_q\left(B\right)}{\varphi_p\left(B\right)} = \frac{\theta_{q_1}\left(B\right)G\left(B\right)}{\varphi_{p_1}\left(B\right)G\left(B\right)} = \frac{\theta_{q_1}\left(B\right)}{\varphi_{p_1}\left(B\right)} \equiv \psi_1(B) \tag{13.29}$$

and

$$W_t = \psi_1(B)u_t. \tag{13.30}$$

(13.26) and (13.27) have the same $MA(\infty)$ representation, hence also the same autocovariance generating functions:

$$\gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2\psi_1(z)\psi_1(z^{-1}). \tag{13.31}$$

It is not possible to distinguish a series generated by (13.26) form one produced with (13.27). Among these two models, we will select the simpler one, $i.e.$ (13.27). Further, if we tried to estimate (13.26) rather than (13.27), we would meet singularity problems (in the covariance matrix of the estimators).

References

ANDERSON, O. D. (1975): "On a Paper by Davies, Pete and Frost Concerning Maximum Autocorrelations for Moving Average Processes," *Australian Journal of Statistics*, 17, 87.

ANDERSON, T. W. (1971): *The Statistical Analysis of Time Series*. John Wiley & Sons, New York.

BOX, G. E. P., AND G. M. JENKINS (1976): *Time Series Analysis: Forecasting and Control*. Holden-Day, San Francisco, second edn.

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods*. Springer-Verlag, New York, second edn.

CHANDA, K. C. (1962): "On Bounds of Serial Correlations," *Annals of Mathematical Statistics*, 33, 1457.

DUFOUR, J.-M. (1999a): "Notions of Asymptotic Theory," Lecture notes, Département de sciences économiques, Université de Montréal.

——— (1999b): "Properties of Moments of Random Variables," Lecture notes, Département de sciences économiques, Université de Montréal.

DURBIN, J. (1960): "Estimation of Parameters in Time Series Regression Models," *Journal of the Royal Statistical Society, Series A*, 22, 139–153.

GOURIÉROUX, C., AND A. MONFORT (1997): *Time Series and Dynamic Models*. Cambridge University Press, Cambridge, U.K.

KENDALL, M., A. STUART, AND J. K. ORD (1983): *The Advanced Theory of Statistics. Volume 3: Design and Analysis and Time Series*. Macmillan, New York, fourth edn.

SPANOS, A. (1999): *Probability Theory and Statistical Inference: Econometric Modelling with Observational Data*. Cambridge University Press, Cambridge, UK.