Multivariate distributions and measures of dependence between random variables *

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1. Random variables

1.1 In general, economic theory specifies exact relations between economic variables. Even a superficial examination of economic data indicates it is not (almost never) possible to find such relationships in actual data. Instead, we have relations of the form:

$$C_t = \alpha + \beta Y_t + \varepsilon_t$$

where ε_t can be interpreted as a "random variable".

1.2 Definition *A random variable* (r.v.) X *is a variable whose behavior can be described by a "probability law". If* X *takes its values in the real numbers, the probability law of* X *can be described by a "distribution function":*

$$F_X(x) = \mathsf{P}\left[X \leq x\right]$$

1.3 If X is continuous, there is a "density function" $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx\ .$$

The mean and variance of X are given by:

$$\mu_X = \mathsf{E}(X) = \int_{-\infty}^{+\infty} x\, dF_X(x) \qquad \text{(general case)}$$

$$= \int_{-\infty}^{+\infty} x\, f_X(x)\, dx \qquad \text{(continuous case)}$$

$$\mathsf{V}(X) = \sigma_X^2 = \mathsf{E}\left[(X-\mu_X)^2\right] = \int_{-\infty}^{+\infty} (x-\mu_X)^2\, dF_X(x) \qquad \text{(general case)}$$

$$= \int_{-\infty}^{+\infty} (x-\mu_X)^2\, F_X(x)\, dx \qquad \text{(continuous case)}$$

$$= \mathsf{E}\left(X^2\right) - \left[\mathsf{E}(X)\right]^2$$

1.4 It is easy to characterize relations between two non-random variables x and

y :

$$g(x,y)=0$$

or (in certain cases)

$$y=f(x)\ .$$

How does one characterize the links or relations between random variables? The behavior of a pair $(X,Y)'$ is described by a joint distribution function:

$$\begin{aligned} F(x,y) &= \mathsf{P}[X\le x,\ Y\le y] \\ &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy \qquad \text{(continuous case.)} \end{aligned}$$

We call $f(x,y)$ the joint density function of $(X,Y)'$. More generally, if we consider k v.a.'s $X_1, X_2, \ldots, X_k$, their behavior can be described through a k-dimensional distribution function:

$$F(x_1,x_2,\ldots,x_k)=\mathsf{P}[X_1\le x_1,X_2\le x_2,\ \ldots,\ X_k\le x_k]$$

$$=\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f(x_1,x_2,\ldots,x_k)\,dx_1dx_2\cdots dx_k \qquad \text{(continuous case)}$$

where $f(x_1,x_2,\ \ldots,\ x_k)$ is the joint density function of $X_1,X_2,\ \ldots,\ X_k$.

2. Covariances and correlations

2.1. Covariance and correlation between two random variables

We often wish to have a simple measure of association between two random variables X and Y. The notions of "covariance" and "correlation" provide such measures of association. Let X and Y be two *r.v.*'s with means μ_X and μ_Y and finite variances σ_X^2 and σ_Y^2. Below *a.s.* means "almost surely" (with probability 1).

2.1 Definition *The covariance between X and Y is defined by*

$$\mathsf{C}(X,Y)\equiv\sigma_{XY}\equiv\mathsf{E}[(X-\mu_X)(Y-\mu_Y)]\ .$$

2.2 Definition *Suppose $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$. Then the correlation between X and Y is defined by*

$$\rho(X,Y) \equiv \rho_{XY} \equiv \sigma_{XY}/\sigma_X\sigma_Y \,.$$

When $\sigma_X^2 = 0$ or $\sigma_Y^2 = 0$, we set $\rho_{XY} = 0$.

2.3 Theorem *The covariance and correlation between X and Y satisfy the following properties:*

(a) $\sigma_{XY} = \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y)$;

(b) $\sigma_{XY} = \sigma_{YX}$, $\rho_{XY} = \rho_{YX}$;

(c) $\sigma_{XX} = \sigma_X^2$, $\rho_{XX} = 1$;

(d) $\sigma_{XY}^2 \leq \sigma_X^2\sigma_Y^2$; (*Cauchy-Schwarz inequality*)

(e) $-1 \leq \rho_{XY} \leq 1$;

(f) *X and Y are independent* $\Rightarrow \sigma_{XY} = 0 \Rightarrow \rho_{XY} = 0$;

(g) *if $\sigma_X^2 \neq 0$ and $\sigma_Y^2 \neq 0$* ,

$$\rho_{XY}^2 = 1 \Leftrightarrow \left[\exists \text{ two constants } a \text{ and } b \text{ such that } a \neq 0 \text{ and } Y = aX + b \text{ a.s.}\right]$$

PROOF (a)

$$\begin{aligned}
\sigma_{XY} &= \mathsf{E}[(X-\mu_X)(Y-\mu_Y)] \\
&= \mathsf{E}[XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y] \\
&= \mathsf{E}(XY) - \mu_X\mathsf{E}(Y) - \mathsf{E}(X)\mu_Y + \mu_X\mu_Y \\
&= \mathsf{E}(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_y \\
&= \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y) \,.
\end{aligned}$$

(b) et (c) are immediate. To get (d), we observe that

$$\mathsf{E}\left\{[Y-\mu_Y-\lambda(X-\mu_X)]^2\right\} = \mathsf{E}\left\{[(Y-\mu_Y)-\lambda(X-\mu_X)]^2\right\}$$

$$= \mathsf{E}\left\{ (Y-\mu_Y)^2 - 2\lambda\,(X-\mu_X)\,(Y-\mu_Y) + \lambda^2\,(X-\mu_X)^2 \right\}$$
$$= \sigma_Y^2 - 2\lambda\,\sigma_{XY} + \lambda^2\sigma_X^2 \geq 0\,.$$

for any arbitrary constant λ. In other words, the second-order polynomial $g(\lambda) = \sigma_Y^2 - 2\lambda\sigma_{XY} + \lambda^2\sigma_X^2$ cannot take negative values. This can happen only if the equation

$$\lambda^2\sigma_X^2 - 2\lambda\sigma_{XY} + \sigma_Y^2 = 0 \tag{2.1}$$

does not have two distinct real roots, i.e. the roots are either complex or identical. The roots of equation (2.1). are given by

$$\lambda = \frac{2\sigma_{XY} \pm \sqrt{4\sigma_{XY}^2 - 4\sigma_X^2\sigma_Y^2}}{2\sigma_X^2} = \frac{\sigma_{XY} \pm \sqrt{\sigma_{XY}^2 - \sigma_X^2\sigma_Y^2}}{\sigma_X^2}\,.$$

Distinct real roots are excluded when $\sigma_{XY}^2 - \sigma_X^2\sigma_Y^2 \leq 0$, hence

$$\sigma_{XY}^2 \leq \sigma_X^2\sigma_Y^2\,.$$

(e)

$$\begin{aligned} \sigma_{XY}^2 \leq \sigma_X^2\sigma_Y^2 \;&\Rightarrow\; -\sigma_X\sigma_Y \leq \sigma_{XY} \leq \sigma_X\sigma_Y \\ &\Rightarrow\; -1 \leq \rho_{XY} \leq 1\,. \end{aligned}$$

(f)

$$\begin{aligned} \sigma_{XY} &= \mathsf{E}\{(X-\mu_X)\,(Y-\mu_Y)\} = \mathsf{E}\,(X-\mu_X)\,\mathsf{E}\,(Y-\mu_Y) \\ &= [\mathsf{E}\,(X)-\mu_X]\,[\mathsf{E}\,(Y)-\mu_Y] = 0\,, \\ \rho_{XY} &= \sigma_{XY}\big/\sigma_X\sigma_Y = 0\,. \end{aligned}$$

Note the reverse implication does not hold in general, *i.e.*,

$$\rho_{XY} = 0 \neq> X \text{ and } Y \text{ are independent}$$

(g) 1) Necessity of the condition. If $Y = aX + b$, then

$$\mathsf{E}(Y) = a\mathsf{E}(X) + b = a\mu_X + b\ ,\ \ \sigma_Y^2 = a^2\sigma_X^2\ ,$$

and

$$\sigma_{XY} = \mathsf{E}\left[(Y-\mu_Y)(X-\mu_X)\right] = \mathsf{E}\left[a(X-\mu_X)(X-\mu_X)\right] = a\sigma_X^2\ .$$

Consequently,

$$\rho_{XY}^2 = \frac{a^2\sigma_X^4}{a^2\sigma_X^2\sigma_X^2} = 1\ .$$

2) Sufficiency of the condition. If $\rho_{XY}^2 = 1$, then

$$\sigma_{XY}^2 - \sigma_X^2\sigma_Y^2 = 0.$$

In this case, the equation

$$\mathsf{E}\left\{\left[(Y-\mu_Y) - \lambda\,(X-\mu_X)\right]^2\right\} = \sigma_Y^2 - 2\lambda\sigma_{XY} + \lambda^2\sigma_X^2 = 0$$

has one and only one root

$$\lambda = \frac{2\sigma_{XY}}{2\sigma_X^2} = \sigma_{XY}/\sigma_X^2\ ,$$

so that

$$\mathsf{E}\left\{\left[(Y-\mu_Y) - \frac{\sigma_{XY}}{\sigma_X^2}(X-\mu_X)\right]^2\right\} = 0$$

and

$$\mathsf{P}\left[(Y-\mu_Y) - \frac{\sigma_{XY}}{\sigma_X^2}(X-\mu_X) = 0\right] = \mathsf{P}\left[Y = \frac{\sigma_{XY}}{\sigma_X^2}X + \left(\mu_Y - \frac{\sigma_{XY}}{\sigma_X^2}\mu_X\right)\right] = 1$$

We can thus write:

$$Y = aX + b \text{ with probability } 1$$

where $a = \sigma_{XY}/\sigma_X^2$ and $b = \mu_Y - \frac{\sigma_{XY}}{\sigma_y^2}\mu_X$. $\square$

2.2. Covariances and correlations between k random variables

Consider now k *r.v.´*s $X_1, X_2, \dots, X_k$ such that

$$\begin{aligned} \mathsf{E}(X_i) &= \mu_i \,, \ i = 1, \dots, k \,, \\ \mathsf{C}(X_i, X_j) &= \sigma_{ij} \,, \ i, j = 1, \dots, k \,. \end{aligned}$$

We often wish to compute the mean and variance of a linear combination of $X_1, \dots, X_k$:

$$\Sigma_{i=1}^{k} a_i X_i = a_1 X_1 + a_2 X_2 + \cdots + a_k X_k \,.$$

It is easily verified that

$$\mathsf{E}\left[\Sigma_{i=1}^{k} a_i X_i\right] = \Sigma_{i=1}^{k} a_i \mu_i$$

and

$$\begin{aligned} \mathsf{V}\left[\Sigma_{i=1}^{k} a_i X_i\right] &= \mathsf{E}\left\{\left[\Sigma_{i=1}^{k} a_i (X_i - \mu_i)\right]\left[\Sigma_{j=1}^{k} a_j \left(X_j - \mu_j\right)\right]\right\} \\ &= \Sigma_{i=1}^{k} \Sigma_{j=1}^{k} a_i a_j \sigma_{ij} \,. \end{aligned}$$

Since such formulae may often become cumbersome, it will be convenient to use vector and matrix notation

We define a random vector $\mathbf{X}$ and its mean value $\mathsf{E}(\mathbf{X})$ by:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} \ , \ \mathsf{E}(\mathbf{X}) = \begin{pmatrix} \mathsf{E}(X_1) \\ \vdots \\ \mathsf{E}(X_k) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} \equiv \mu_X \,.$$

Similarly, we define a random matrix M and its mean value $\mathsf{E}(M)$ by:

$$M = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mn} \end{bmatrix} \,, \ \mathsf{E}(M) = \begin{bmatrix} \mathsf{E}(X_{11}) & \mathsf{E}(X_{12}) & \dots & \mathsf{E}(X_{1n}) \\ \mathsf{E}(X_{21}) & \mathsf{E}(X_{22}) & \dots & \mathsf{E}(X_{2n}) \\ \vdots & \vdots & & \vdots \\ \mathsf{E}(X_{m1}) & \mathsf{E}(X_{m2}) & \dots & \mathsf{E}(X_{mn}) \end{bmatrix}$$

where the X_{ij} are *r.v.´*s. To a random vector $\mathbf{X}$, we can associate a covariance matrix $\mathsf{V}(\mathbf{X})$:

$$
\begin{aligned}
\mathsf{V}(\mathbf{X}) &= \mathsf{E}\left\{[\mathbf{X}-\mathsf{E}(\mathbf{X})]\,[\mathbf{X}-\mathsf{E}(\mathbf{X})]'\right\} = \mathsf{E}\left\{[\mathbf{X}-\mu_X]\,[\mathbf{X}-\mu_X]'\right\} \\
&= \mathsf{E}\left\{\begin{bmatrix} (X_1-\mu_1)(X_1-\mu_1) & (X_1-\mu_1)(X_2-\mu_2) & \dots & (X_1-\mu_1)(X_k-\mu_k) \\ \vdots & \vdots & & \vdots \\ (X_k-\mu_k)(X_1-\mu_1) & (X_k-\mu_k)(X_2-\mu_2) & \dots & (X_k-\mu_k)(X_k-\mu_k) \end{bmatrix}\right\} \\
&= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1k} \\ \vdots & \vdots & & \vdots \\ \sigma_{k1} & \sigma_{k2} & \dots & \sigma_{kk} \end{bmatrix} = \Sigma\,.
\end{aligned}
$$

If $\mathbf{a} = (a_1, \dots, a_k)'$, we see that:

$$\Sigma_{i=1}^{k} a_i X_i = \mathbf{a}'\mathbf{X}\,.$$

Basic properties of $\mathsf{E}(\mathbf{X})$ and $\mathsf{V}(\mathbf{X})$ are summarized by the following proposition.

2.4 Proposition *Let* $\mathbf{X} = (X_1, \dots, X_k)'$ *a* $k \times 1$ *random vector,* α *a scalar,* $\mathbf{a}$ *and* $\mathbf{b}$ *fixed* $k \times 1$ *vectors, and* A *a fixed* $g \times k$ *matrix. Then, provided the moments considered are finite, we have the following properties:*

(a) $\mathsf{E}(\mathbf{X}+\mathbf{a}) = \mathsf{E}(\mathbf{X})+\mathbf{a}$;

(b) $\mathsf{E}(\alpha\mathbf{X}) = \alpha\mathsf{E}(\mathbf{X})$;

(c) $\mathsf{E}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{E}(\mathbf{X})$, $\mathsf{E}(A\mathbf{X}) = A\mathsf{E}(\mathbf{X})$;

(d) $\mathsf{V}(\mathbf{X}+\mathbf{a}) = \mathsf{V}(\mathbf{X})$;

(e) $\mathsf{V}(\alpha\mathbf{X}) = \alpha^2\mathsf{V}(\mathbf{X})$;

(f) $\mathsf{V}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$, $\mathsf{V}(A\mathbf{X}) = A\mathsf{V}(\mathbf{X})\,A'$;

(g) $\mathsf{C}(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{a}'\mathsf{V}(\mathbf{X})\,\mathbf{b} = \mathbf{b}'\mathsf{V}(\mathbf{X})\,\mathbf{a}$.

2.5 Theorem *Let* $\mathbf{X} = (X_1, \dots, X_k)'$ *be a random vector with covariance matrix* $\mathsf{V}(\mathbf{X}) = \Sigma$. *Then we have the following properties:*

(a) $\Sigma' = \Sigma$;

(b) Σ *is a positive semidefinite matrix;*

(c) $0 \leq |\Sigma| \leq \sigma_1^2 \sigma_2^2 \ldots \sigma_k^2$ *where* $\sigma_i^2 = \mathsf{V}(X_i)\,,\;\; i = 1,\ldots,k$;

(d) $|\Sigma| = 0 \Leftrightarrow$ *there is at least one linear relation between the* r.v.´s $X_1,\ldots,X_k$*, i.e., we can find constants* $a_1,\ldots,a_k$*,* b *not all equal to zero such that* $a_1 X_1 + \cdots + a_k X_k = b$ *with probability 1;*

(e) *rank*$(\Sigma) = r < k \Leftrightarrow \mathbf{X}$ *can be expressed in the form*

$$\mathbf{X} = B\mathbf{Y} + \mathbf{c}$$

where $\mathbf{Y}$ *is a random vector of dimension* r *whose covariance matrix is* I_r, B *is a* $k \times r$ *matrix of rank* r*, and* $\mathbf{c}$ *is a* $k \times 1$ *constant vector.*

2.6 Remark We call the determinant $|\Sigma|$ the *generalized variance of* $\mathbf{X}$.

2.7 Definition *If we consider two random vectors* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *with dimensions* $k_1 \times 1$ *and* $k_2 \times 1$ *respectively, the covariance matrix between* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *is defined by:*

$$\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) = \mathsf{E}\left\{[\mathbf{X}_1 - \mathsf{E}(\mathbf{X}_1)]\,[\mathbf{X}_2 - \mathsf{E}(\mathbf{X}_2)]'\right\} \,.$$

The following proposition summarizes some basic properties of $\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2)$.

2.8 Proposition *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *two random vectors of dimensions* $k_1 \times 1$ *and* $k_2 \times 1$ *respectively. Then, provided the moments considered are finite we have the following properties:*

(a) $\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) = \mathsf{E}[\mathbf{X}_1 \mathbf{X}_2'] - \mathsf{E}(\mathbf{X}_1)\,\mathsf{E}(\mathbf{X}_2)'$;

(b) $\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) = \mathsf{C}(\mathbf{X}_2, \mathbf{X}_1)'$;

(c) $\mathsf{C}(\mathbf{X}_1, \mathbf{X}_1) = \mathsf{V}(\mathbf{X}_1)\,,\;\; \mathsf{C}(\mathbf{X}_2, \mathbf{X}_2) = \mathsf{V}(\mathbf{X}_2)$;

(d) *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are fixed vectors of dimensions* $k_1 \times 1$ *and* $k_2 \times 1$ *respectively,*

$$\mathsf{C}(\mathbf{X}_1 + \mathbf{a}, \mathbf{X}_2 + \mathbf{b}) = \mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) \;;$$

(e) *if α and β are two scalar constants,*

$$\mathsf{C}(\alpha \mathbf{X}_1, \beta \mathbf{X}_2) = \alpha\beta\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) \ ;$$

(f) *if $\mathbf{a}$ and $\mathbf{b}$ are fixed $k_1 \times 1$ and $k_2 \times 1$ vectors,*

$$\mathsf{C}(\mathbf{a}'\mathbf{X}_1, \mathbf{b}'\mathbf{X}_2) = \mathbf{a}'\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2)\,\mathbf{b} \ ;$$

(g) *if A and B are fixed matrices matrices with dimensions $g_1 \times k_1$ and $g_2 \times k_2$ respectively,*

$$\mathsf{C}(A\mathbf{X}_1, B\mathbf{X}_2) = \mathbf{A}\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2)\,\mathbf{B}' \ ;$$

(h) *if $k_1 = k_2$ and $\mathbf{X}_3$ is a $k \times 1$ random vector,*

$$\mathsf{C}(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{X}_3) = \mathsf{C}(\mathbf{X}_1, \mathbf{X}_3) + \mathsf{C}(\mathbf{X}_2, \mathbf{X}_3) \ ;$$

(i) *if $k_1 = k_2$,*

$$\begin{aligned} \mathsf{V}(\mathbf{X}_1 + \mathbf{X}_2) &= \mathsf{V}(\mathbf{X}_1) + \mathsf{V}(\mathbf{X}_2) + \mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) + \mathsf{C}(\mathbf{X}_2, \mathbf{X}_1) \ , \\ \mathsf{V}(\mathbf{X}_1 - \mathbf{X}_2) &= \mathsf{V}(\mathbf{X}_1) + \mathsf{V}(\mathbf{X}_2) - \mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) - \mathsf{C}(\mathbf{X}_2, \mathbf{X}_1) \ . \end{aligned}$$

3. Multinormal distribution

Consider two random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ with dimensions $k_1 \times 1$ and $k_2 \times 1$ respectively. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent, then

$$\mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) \equiv \mathsf{E}\left[\left(\mathbf{X}_1 - \mu_{X_1}\right)\left(\mathbf{X}_2 - \mu_{X_2}\right)'\right] = 0$$

The reverse implication is not true in general, except in special cases. One such case is the one where the random vector $\mathbf{X} = \left(\mathbf{X}_1', \mathbf{X}_2'\right)'$ follows a multinormal distribution.

3.1 Definition *We say that the $k \times 1$ random vector $\mathbf{X}$ follows a multinormal distribution with mean μ and covariance matrix Σ, denoted $\mathbf{X} \sim N_k[\mu, \Sigma]$, if the* *characteristic function of* $\mathbf{X}$ *has the form:*

$$\mathsf{E}\left[e^{i\mathbf{t}'\mathbf{X}}\right] = e^{i\mu'\mathbf{t}-\frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}}, \quad \mathbf{t} \in \mathscr{R}^k,\ i = \sqrt{-1}\,.$$

3.2 When $|\Sigma| \neq 0$, the vector $\mathbf{X}$ has a density function of the form:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right]$$

If $k = 1$, then $\Sigma = \sigma^2$ and

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}(x-\mu)\frac{1}{\sigma^2}(x-\mu)\right] = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]\,.$$

Some important properties of the multinormal distribution are summarized in the following theorem.

3.3 Theorem *If* $\mathbf{X} \sim N_k[\mu, \Sigma]$, *then*

(a) $\mathbf{X} + \mathbf{c} \sim N_k[\mu + \mathbf{c}, \Sigma]$, *for any fixed* $k \times 1$ *vector* $\mathbf{c}$;

(b) $\mathbf{a}'\mathbf{X} \sim N_1[\mathbf{a}'\mu, \mathbf{a}'\Sigma\mathbf{a}]$, *for any fixed* $k \times 1$ *vector* $\mathbf{a}$;

(c) $A\mathbf{X} \sim N_g[A\mu, A\Sigma A']$, *for any fixed* $g \times k$ *matrix* A ;

(d) *if*

$$\mathbf{X} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix} \sim N_k\left[\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\,,\,\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\right],$$

where $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are vectors of dimensions* $k_1 \times 1$ *and* $k_2 \times 1$,

$$\begin{aligned}\mu_1 &= \mathsf{E}(\mathbf{X}_1), \mu_2 = \mathsf{E}(\mathbf{X}_2),\ \Sigma_{11} = \mathsf{C}(\mathbf{X}_1, \mathbf{X}_1),\ \Sigma_{22} = \mathsf{C}(\mathbf{X}_2, \mathbf{X}_2),\\ \Sigma_{12} &= \mathsf{C}(\mathbf{X}_1, \mathbf{X}_2) = \Sigma_{21}',\end{aligned}$$

then

(i) $\mathbf{X}_1 \sim N_{k_1}[\mu_1, \Sigma_{11}]$, $\mathbf{X}_2 \sim N_{k_2}[\mu_2, \Sigma_{22}]$;

(ii) $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent* $\Leftrightarrow \Sigma_{12} = 0$;

(iii) *the conditional distribution of* $\mathbf{X}_2$ *given* $\mathbf{X}_1$ *is normal with mean and et variance*

$$\begin{aligned} \mathsf{E}[\mathbf{X}_2 | \mathbf{X}_1] &= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{X}_1 - \mu_1) , \\ \mathsf{V}[\mathbf{X}_2 | \mathbf{X}_1] &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} , \end{aligned}$$

i.e.

$$\mathbf{X}_2 | \mathbf{X}_1 \sim N_{k_2}\left[\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{X}_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right] .$$

3.4 Theorem *If* $\mathbf{X} \sim N_k[\mu, \Sigma]$ *with* $|\Sigma| \neq 0$, *then*

$$(\mathbf{X} - \mu)' \Sigma^{-1} (\mathbf{X} - \mu) \sim \chi^2(k) .$$

PROOF Since Σ is a positive definite matrix ($|\Sigma| \neq 0$), there exists a nonsingular matrix P such that

$$P\Sigma P' = I_k$$

hence

$$\begin{aligned} \Sigma &= P^{-1}(P')^{-1} = (P'P)^{-1} , \\ \Sigma^{-1} &= P'P . \end{aligned}$$

Consequently,

$$\begin{aligned} (\mathbf{X} - \mu)' \Sigma^{-1} (\mathbf{X} - \mu) &= (\mathbf{X} - \mu)' P'P (\mathbf{X} - \mu) \\ &= [P(\mathbf{X} - \mu)]' [P(\mathbf{X} - \mu)] = \mathbf{v}'\mathbf{v} = \Sigma_{i=1}^k v_i^2 \end{aligned}$$

where

$$\mathbf{v} \equiv P[\mathbf{X} - \mu] = (v_1, v_2, \ldots, v_k)' .$$

Since $\mathbf{X} \sim N[\mu, \Sigma]$, we have $\mathbf{X} - \mu \sim N[\mathbf{0}, \Sigma]$, hence

$$P[\mathbf{X} - \mu] \sim N[\mathbf{0}, P\Sigma P'] ,$$

and

$$\mathbf{v} = P[\mathbf{X} - \mu] \sim N[\mathbf{0}, I_k] .$$

Thus $v_1, \ldots, v_k$ are i.i.d. $N[0,1]$ and $(\mathbf{X}-\mu)' \Sigma^{-1} (\mathbf{X}-\mu) = \Sigma_{i=1}^{k} v_i^2 \sim \chi^2(k)$. $\square$