Sequences and series *

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First version: March 1992
Revised: January 2002, October 2016
This version: October 2016
Compiled: January 10, 2017, 15:36

* This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), the Toulouse School of Economics (Pierre-de-Fermat Chair of excellence), the Universitad Carlos III de Madrid (Banco Santander de Madrid Chair of excellence), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

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1. Definitions and notation

1.1 Notation *We shall use the following notation:*

(1) *iff : if and only if;*

(2) $\Leftrightarrow$ *: if and only if;*

(3) ∞ : *infinity ;*

(4) A^c *: complement of the set* A*;*

(5) $\Rightarrow$ *: implies ;*

(6) $\sim$ *: is distributed like;*

(7) $\equiv$ *: equal by definition;*

(8) $\mathbb{C}$ *: set of complex numbers;*

(9) $\mathbb{R}$ *: real numbers;*

(10) $\mathbb{Z}$ *: integers;*

(11) $\mathbb{N}_0 = \{0,\ 1,\ 2,\ ...\}$: *nonnegative integers;*

(12) $\mathbb{N} = \{1,\ 2,\ 3,\ ...\}$ *: positive integers;*

(13) $\overline{\mathbb{R}}$ *: extended real numbers :*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}\ .$$

1.2 Notation **1.3 Definition** BOUNDED SET IN $\mathbb{R}$. *Let $E \subseteq \mathbb{R}$. If there is an element $y \in \mathbb{R}$ such that $x \leq y$, $\forall x \in \mathbb{R}$, we say that E has an upper bound (or is bounded from above). If there is an element $z \in \mathbb{R}$ such that $x \geq z$, $\forall x \in E$, we say that E has a lower bound (or is bounded from below). If E has both upper and lower bounds, we say that E is bounded.*

1.4 Definition SUPREMUM AND INFIMUM. *Let $E \subseteq \overline{\mathbb{R}}$. $\sup(E)$ is the smallest element of $\overline{\mathbb{R}}$ such that $x \leq \sup(E)$, $\forall x \in E$; $\inf(E)$ is the largest element of $\overline{\mathbb{R}}$ such that $\inf(E) \leq x$, $\forall x \in E$.*

1.5 Definition BOUNDED SET IN $\mathbb{C}$. *Let $E \subseteq \mathbb{C}$. If there is a real number M and a complex number z_0 such that $|z - z_0| < M$ for all $z \in E$, we say the set E is bounded.*

1.6 Definition SEQUENCE. *Let E be a set. A sequence in E is function $f(n) = a_n$ which associates to each element $n \in \mathbb{N}$ an element $a_n \in E$. The sequence is usually denoted by the ordered set of the values of $f(n)$:*

$$\{a_1, a_2, \ldots\} \equiv \{a_n\}_{n=1}^{\infty} \equiv \{a_n\}$$

or

$$(a_1, a_2, \ldots) \equiv (a_n)_{n=1}^{\infty} \equiv (a_n) \ .$$

If $E = \mathbb{C}$, the sequence is complex. If $E = \mathbb{R}$, the sequence is real. To indicate that all the elements of the sequence $\{a_n\}$ are in E, we write $\{a_n\} \subseteq E$.

1.7 Remark Let $m \in \mathbb{Z}$ and $I_m = \{n \in \mathbb{Z} : n \geq m\}$. A function $f(n) = b_n$ which maps every element $n \in I_m$ to an element $a_n \in E$ can be viewed as a sequence in E on defining $a_n = b_{m+n-1}$, $n = 1, 2, \ldots$ Such a sequence is usually denoted

$$\{b_m, b_{m+1}, \ldots\} \equiv \{b_n\}_{n=m}^{\infty} \ .$$

Similarly, if $I_m = \{n \in \mathbb{Z} : n \leq m\}$, we can define $a_n = b_{m-n+1}$, $n = 1, 2, \ldots$. In this case, the sequence can be denoted as

$$\{\ldots, b_{m-1}, b_m\} \equiv \{b_n\}_{n=-\infty}^{m} \ .$$

1.8 Definition SUBSEQUENCE. *Let E be a set, $\{a_n\}_{n=1}^{\infty} \subseteq E$, and $\{n_k\}_{k=1}^{\infty}$ a sequence of positive integers such that $n_1 < n_2 < \cdots$. The sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.*

1.9 Definition LIMIT OF A COMPLEX SEQUENCE. *Let $a \in \mathbb{C}$ and $\{a_n\} \subseteq \mathbb{C}$. The sequence $\{a_n\}$ converges to a iff for any real number $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies $|a_n - a| < \varepsilon$. In this case, we write $a_n \to a$, or*

$$\lim_{n\to\infty} a_n = a \ ,$$

and a is called the limit of $\{a_n\}$. If there is a number $a \in \mathbb{C}$ such that $a_n \to a$, we say that the sequence $\{a_n\}$ converges (or converges in $\mathbb{C}$). If the sequence does not converge, we say it diverges.

1.10 Remark When there is no ambiguity, we can also write $\lim\ a_n$ instead of $\lim_{n\to\infty} a_n$.

1.11 Definition CONVERGENCE IN A SET. *Let $E \subseteq \mathbb{C}$ and $\{a_n\} \subseteq E$. If there exists an element $a \in E$ such that $a_n \to a$, we say that $\{a_n\}$ converges in E.*

1.12 Definition CONVERGENCE IN THE SENSE OF CAUCHY. *Let $\{a_n\} \subseteq \mathbb{C}$. The sequence $\{a_n\}$ converges in the Cauchy sense iff for any $\varepsilon > 0$, there exists an integer N such that $m \geq N$ and $n \geq N$ imply $|a_m - a_n| < \varepsilon$. A sequence which converges in the Cauchy sense is called a Cauchy sequence.*

1.13 Definition INFINITE LIMITS. *Let $\{a_n\} \subseteq \mathbb{R}$. We say that the sequence $\{a_n\}$ diverges to ∞ iff for any real number M there exists an integer N such that $n \geq N$ implies $a_n \geq M$. In this case, we write $a_n \to \infty$ or*

$$\lim_{n\to\infty} a_n = \infty \ .$$

Similarly, we say the sequence $\{a_n\}$ diverges to $-\infty$ iff for any real number M there is an integer N such that $n \geq N$ implies $a_n \leq M$. In this case, we write $a_n \to -\infty$ or

$$\lim_{n\to\infty} a_n = -\infty \ .$$

We also wrote $+\infty$ instead of ∞.

1.14 Definition MONOTONIC SEQUENCE. *Let $\{a_n\} \subseteq \mathbb{R}$. If $a_n \leq a_{n+1}$, for all $n \in \mathbb{N}$, we say that the sequence $\{a_n\}$ is monotonically increasing (or monotonic increasing) . If $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, we say the sequence $\{a_n\}$ is monotonically decreasing (monotonic decreasing). If $\{a_n\}$ is monotonically increasing and $a_n \to a$, we write $a_n \uparrow a$. If $\{a_n\}$ is monotonically decreasing and $a_n \to a$, we write $a_n \downarrow a$.*

1.15 Definition UPPER AND LOWER LIMITS. *Let $\{a_n\} \subseteq \mathbb{R}$. The upper limit of the sequence $\{a_n\}$ is defined by*

$$\limsup_{n\to\infty} a_n = \inf_{N\geq 1}\left\{\sup_{n\geq N} a_n\right\} \equiv \inf\{\sup\{a_n : n \geq N\} : N \geq 1\} .$$

The lower limit of the sequence $\{a_n\}$is defined by

$$\liminf_{n\to\infty} a_n = \sup_{N\geq 1}\left\{\inf_{n\geq N} a_n\right\} \equiv \sup\{\inf\{a_n : n \geq N\} : N \geq 1\} .$$

We also write $\overline{\lim}$ instead of $\limsup$, and $\underline{\lim}$ instead of $\liminf$.

1.16 Remark The upper and lower limits of the sequence $\{a_n\} \subseteq \mathbb{R}$ always exist in $\overline{\mathbb{R}}$.

1.17 Definition ACCUMULATION POINT. *Let $\{a_n\} \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. a is an accumulation point of $\{a_n\}$ iff for any real number $\varepsilon > 0$, the inequality $|a_n - a| < \varepsilon$ is satisfied for an infinity of elements of the sequence $\{a_n\}$.*

1.18 Definition PARTIAL SUM AND SERIES. *Let $\{a_n\} \subseteq \mathbb{C}$ and $S_N = \sum_{n=1}^{N} a_n$. We call $\{S_N\}_{N=1}^{\infty}$ the sequence of partial sums associated with $\{a_n\}$. The symbol $\sum_{n=1}^{\infty} a_n$ represents the series associated with $\{a_n\}$. If $\lim_{N\to\infty} S_N = S$ where $S \in \mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ converges (or converges to S) and we write*

$$\sum_{n=1}^{\infty} a_n = S .$$

If the series $\sum_{n=1}^{\infty} a_n$ does not converge, we say it diverges.

1.19 Remark If we consider a sequence of the form $\{a_n\}_{n=m}^{\infty}$ where $m \in \mathbb{Z}$, we say that the series $\sum_{n=m}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$, where $S_N = \sum_{n=m}^{N+(m-1)} a_n$. Similarly, for a sequence of the form $\{a_n\}_{n=-\infty}^{m}$, where $m \in \mathbb{Z}$, we say that the series $\sum_{n=-\infty}^{m} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$, where $S_N = \sum_{n=m}^{m+1-N} a_n$.

1.20 Definition ABSOLUTE AND CONDITIONAL CONVERGENCE. *Let $\{a_n\} \subseteq \mathbb{C}$. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, we say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.*

1.21 Definition TWO-SIDED SEQUENCE. *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{a_n\}_{n=-\infty}^{-1}$ *be two sequences of complex numbers. If the series* $\sum_{n=0}^{\infty} a_n$ *converges to* $S_1 \in \mathbb{C}$ *and if the series* $\sum_{n=-\infty}^{-1} a_n$ *converges to* $S_2 \in \mathbb{C}$, *we say that the two-sided series* $\sum_{n=-\infty}^{\infty} a_n$ *converges to* $S_1 + S_2$.

1.22 Definition DOUBLE SEQUENCE. *A double sequence in* E *is a function* $f(m,\ n) = a_{mn}$ *which maps each pair* $(m,n) \in \mathbb{N}^2$ *to an element* $a_{mn} \in E$. *We usually denote the double sequence by*

$$\{a_{mn}\}_{m,n=1}^{\infty} \equiv \{a_{mn}\} \ .$$

To indicate that all the elements of the double sequence $\{a_{mn}\}$ *are in* E, *we write* $\{a_{mn}\} \subseteq E$.

1.23 Definition LIMIT OF A COMPLEX DOUBLE SEQUENCE. *Let* $a \in \mathbb{C}$ *and* $\{a_{mn}\} \subseteq \mathbb{C}$. *The double sequence* $\{a_{mn}\}$ *converges to* a *when* $m,\ n \to \infty$ *iff for any real number* $\varepsilon > 0$, *there is an integer* N *such that* $m,\ n \geq N$ *implies* $|a_{mn} - a| < \varepsilon$. *In this case, we write* $a_{mn} \underset{m,n\to\infty}{\longrightarrow} a$, *or*

$$\lim_{m,n\to\infty} a_{mn} = a \ ,$$

and a *is called the limit of* $\{a_{mn}\}$ *when* $m,\ n \to \infty$.

1.24 Remark For double sequences, we can consider several different limits: $\lim_{m\to\infty} a_{mn}$, $\lim_{n\to\infty} a_{mn}$, $\lim_{m\to\infty}\left[\lim_{n\to\infty} a_{mn}\right]$, $\lim_{n\to\infty}\left[\lim_{m\to\infty} a_{mn}\right]$. In general, these limits are not equal. Even if

$$\lim_{m\to\infty} a_{mn} \equiv b_n \quad , \quad \lim_{n\to\infty} a_{mn} = c_m$$

exist, we *can* have

$$\lim_{n\to\infty}\left[\lim_{m\to\infty} a_{mn}\right] \equiv \lim_{n\to\infty} b_n \neq \lim_{m\to\infty} c_m \equiv \lim_{m\to\infty}\left[\lim_{n\to\infty} a_{mn}\right] \ .$$

2. Convergence of sequences

2.1 Theorem CONVERGENCE CRITERION FOR COMPLEX SEQUENCES. *Let* $\{c_n\} \subseteq \mathbb{C}$, *where* $c_n = a_n + i\, b_n$, $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$, *for all* n, *and* $i = \sqrt{-1}$. *Then the sequence* $\{c_n\}$ *converges iff each one of the sequences* $\{a_n\}$ *and* $\{b_n\}$ *converges. Furthermore, if* $\{c_n\}$ *converges, we have:*

$$\lim_{n\to\infty} c_n = \left(\lim_{n\to\infty} a_n\right) + i\left(\lim_{n\to\infty} b_n\right) \ .$$

2.2 Theorem PROPERTIES OF CONVERGENT SEQUENCES. *Let* $\{a_n\} \subseteq \mathbb{C}$, $a \in \mathbb{C}$ *and* $a' \in \mathbb{C}$.

(a) ***Limit unicity***. *If* $a_n \to a$ *and* $a_n \to a'$, *then* $a = a'$.

(b) ***Boundedness of convergent sequences***. *If the sequence* $\{a_n\}$ *converges, then the set* $\{a_1,\ a_2,\ \ldots\}$ *is bounded.*

(c) ***Convergence of bounded sequences (Bolzano-Weierstrass)***. *If the sequence $\{a_n\}$ is bounded, then $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$. In other words, a bounded sequence $\{a_n\}$ has at least one accumulation point.*

(d) ***Convergence of subsequences***. *$\{a_n\}$ converges to $a \Leftrightarrow$ each subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to $a \Leftrightarrow$ each subsequence $\{a_{n_k}\}$ of $\{a_n\}$ contains another subsequence $\{a_{m_k}\}$ which converges to a.*

(e) ***Accumulation and convergence***. *If the sequence $\{a_n\}$ is bounded and has exactly one accumulation point a, then $a_n \to a$. If the sequence $\{a_n\}$ has no finite accumulation point or has several, then it diverges.*

2.3 Theorem CONVERGENCE OF TRANSFORMED SEQUENCES. *Let $\{a_n\} \subseteq \mathbb{C}$ and $\{b_n\} \subseteq \mathbb{C}$ two sequences such that*

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b$$

where $a, b \in \mathbb{C}$. Then

(a) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(b) $\lim_{n\to\infty}(c\, a_n) = c\, a$, $\lim_{n\to\infty}(a_n + c) = a + c$, $\forall c \in \mathbb{C}$;

(c) $\lim_{n\to\infty}(a_n b_n) = a\, b$;

(d) $\lim(a_n/b_n) = a/b$, provided $b \neq 0$;

(e) $\lim_{n\to\infty} g(a_n) = g(a)$ *for any function* $g : \mathbb{C} \to \mathbb{C}$ *continuous at* $x = a$.

2.4 Theorem CONVERGENCE OF CAUCHY SEQUENCES. *Let $\{a_n\} \subseteq \mathbb{C}$.*

(a) *If the sequence $\{a_n\}$ converges, then $\{a_n\}$ converges in the Cauchy sense.*

(b) ***Completeness***. *If the sequence $\{a_n\}$ converges in the Cauchy sense, then the sequence $\{a_n\}$ converges.*

2.5 Theorem CONVERGENCE OF REAL SEQUENCES. *Let $\{a_n\} \subseteq \mathbb{R}$, $\{b_n\} \subseteq \mathbb{R}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}$.*

(a) $\liminf\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} a_n$.

(b) $\lim\limits_{n\to\infty} a_n = a \Leftrightarrow \liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = a$.

(c) *If $a_n \leq b_n$ for $n \geq N$, then*

$$\begin{aligned} \liminf_{n\to\infty} a_n &\leq \liminf_{n\to\infty} b_n \,, \\ \limsup_{n\to\infty} a_n &\leq \limsup_{n\to\infty} b_n \,. \end{aligned}$$

(d) *If $a_n \leq b_n$ for $n \geq N$ and if $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \,.$$

(e) *If the sequence $\{a_n\}$ is monotonically increasing, then*

$$\{a_n\} \text{ converges in } \mathbb{R} \text{ or } \lim_{n\to\infty} a_n = \infty \,.$$

(f) *If the sequence $\{a_n\}$ is monotonically decreasing, then*

$$\{a_n\} \text{ converges in } \mathbb{R} \text{ or } \lim_{n\to\infty} a_n = -\infty \,.$$

(g) *If the sequence $\{a_n\}$ is monotonic (increasing or decreasing) and bounded, then $\{a_n\}$ converges in $\mathbb{R}$.*

2.6 Theorem Limits of important special sequences. *Let p and α be real numbers and x a complex number.*

(a) *If $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*

(b) *If $p > 0$, $\lim_{n\to\infty} p^{1/n} = 1$.*

(c) $\lim_{n\to\infty} n^{1/n} = 1$.

(d) *If $p > 0$, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.*

(e) *If $|x| < 1$, $\lim_{n\to\infty} x^n = 0$.*

(f) *If $b > 0$ and $b \neq 1$, $\lim_{n\to\infty} [\log_b(n)/n] = 0$.*

3. Convergence of series

3.1 Theorem Cauchy criterion for convergence of a series. *Let $\{a_n\} \subseteq \mathbb{C}$. The series $\sum_{n=1}^{\infty} a_n$ converges iff, for any $\varepsilon > 0$, there is an integer N such that $n \geq m \geq N$ implies $|\sum_{k=m}^{n} a_k| < \varepsilon$.*

3.2 Proposition Alternative forms of the Cauchy criterion for series. *Let $\{a_n\} \subseteq \mathbb{C}$. The series $\sum_{n=1}^{\infty} a_n$ converges*

$\Leftrightarrow$ *for any $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies $\left|\sum_{k=n+1}^{n+p} a_n\right| < \varepsilon$ for any $p \geq 1$*

$\Leftrightarrow$ *for any $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies $\sup_{p\geq 1} \left|\sum_{k=n+1}^{n+p} a_n\right| < \varepsilon$*

$$\Leftrightarrow \quad \lim_{n\to\infty}\left[\sup_{p\geq 1}\left|\sum_{k=n+1}^{n+p} a_n\right|\right] = 0\ .$$

3.3 Theorem NECESSARY CONDITIONS FOR CONVERGENCE OF A SERIES. *Let* $\{a_n\} \subseteq \mathbb{C}$.

(a) *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim\limits_{n\to\infty} a_n = 0$.

(b) *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges and if* $|a_{n+1}| \leq |a_n|$ *for* $n \geq N$, *then* $\lim\limits_{n\to\infty}(na_n) = 0$.

3.4 Corollary DIVERGENCE CRITERION FOR A SERIES. *Let* $\{a_n\} \subseteq \mathbb{C}$ *and* $c > 0$. *If* $|a_n| \geq c$ *for an infinite number values of* n, *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

3.5 Theorem CHARACTERIZATION OF ABSOLUTE CONVERGENCE. *Let* $\{a_n\} \subseteq \mathbb{C}$. *The series* $\sum_{n=1}^{\infty} a_n$ *does not converge absolutely* $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ *diverges* $\Leftrightarrow \sum_{n=1}^{\infty} |a_n| = \infty$.

3.6 Remark To indicate that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, we *can* write $\sum_{n=1}^{\infty} |a_n| < \infty$.

3.7 Theorem CRITERION FOR ABSOLUTE CONVERGENCE OF A SERIES. *Let* $\{a_n\} \subseteq \mathbb{C}$. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

3.8 Corollary DIVERGENCE CRITERION FOR A SERIES. *Let* $\{a_n\} \subseteq \mathbb{C}$. *If the series* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} |a_n| = \infty$.

3.9 Theorem COMPARISON CRITERION FOR CONVERGENCE OF A SERIES. *Let* $\{a_n\} \subseteq \mathbb{C}$, $\{c_n\} \subseteq \mathbb{R}$ *and* $\{d_n\} \subseteq \mathbb{R}$.

(a) *If* $|a_n| \leq c_n$ *for* $n \geq n_0$, *where* n_0 *is a given integer, and if the series* $\sum_{n=1}^{\infty} c_n$ *converges, then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(b) *If* $|a_n| \geq d_n \geq 0$ *for* $n \geq n_0$, *where* n_0 *is a given integer, and if* $\sum_{n=1}^{\infty} d_n$ *diverges, then* $\sum_{n=1}^{\infty} |a_n| = \infty$ *but* $\sum_{n=1}^{\infty} a_n$ *can converge or diverge.*

3.10 Theorem CONVERGENCE OF A SERIES OF NONNEGATIVE NUMBERS. *Let* $\{a_n\} \subseteq \mathbb{R}$ *where* $a_n \geq 0$ *for all* n.

(a) *The series* $\sum_{n=1}^{\infty} a_n$ *converges iff the sequence of partial sums* $\{\sum_{n=1}^{N} a_n\}_{N=1}^{\infty}$ *is bounded.*

(b) ***Cauchy's condensation criterion****. If the sequence* $\{a_n\}$ *is monotonically decreasing* $(a_1 \geq a_2 \geq a_3 \geq \ldots)$, *the series* $\sum_{n=1}^{\infty} a_n$ *converges iff the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2\,a_2 + 4\,a_4 + 8\;a_8 + \ldots \tag{3.1}$$

converges.

3.11 Proposition CONVERGENCE OF SPECIAL SERIES. *Let x a complex number and p a real number.*

(a) ***Geometric series.*** *If $|x| < 1$,*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

where $0^0 \equiv 1$. If $|x| \geq 1$, the series $\sum_{n=0}^{\infty} x^n$ diverges.

(b) *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.*

(c) *The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

(d) $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

3.12 Theorem ROOT CONVERGENCE CRITERION (CAUCHY). *Let $\{a_n\} \subseteq \mathbb{C}$ and $\alpha = \limsup\limits_{n\to\infty} |a_n|^{1/n}$.*

(a) *If $\alpha < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(b) *If $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.*

(c) *If $|a_n|^{1/n} \leq \delta < 1$ for $n \geq n_0$, where n_0 is a given integer, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(d) *If $|a_n|^{1/n} \geq 1$ for an infinite number of values of n, $\sum_{n=1}^{\infty} a_n$ diverges.*

(e) *If none of the preceding conditions holds, we can find cases where $\sum_{n=1}^{\infty} a_n$ converges and cases where $\sum_{n=1}^{\infty} a_n$ diverges.*

3.13 Theorem RATIO CONVERGENCE CRITERION (D'ALEMBERT). *Let $\{a_n\} \subseteq \mathbb{C}$ and $0/0 \equiv 0$.*

(a) *If $\limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(b) *If $\left|\frac{a_{n+1}}{a_n}\right| \leq \varepsilon < 1$ for $n \geq n_0$, where n_0 is a given integer, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(c) *If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$, where n_0 is a given integer, the series $\sum_{n=1}^{\infty} a_n$ diverges.*

(d) *If $\liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.*

(e) *If $\liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, the series $\sum_{n=1}^{\infty} a_n$ can converge or diverge.*

3.14 Theorem RELATION BETWEEN THE ROOT AND RATIO CONVERGENCE TESTS. *Let $\{a_n\} \subseteq \mathbb{C}$ a sequence such that $a_n \neq 0$ for $n \geq n_0$, where n_0 is a given integer. Then*

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty}|a_n|^{1/n} \leq \limsup_{n\to\infty}|a_n|^{1/n} \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| .$$

If we define $0/0 \equiv 0$ and $|x/0| = \infty$ for $x \neq 0$, the above inequalities hold for any sequence $\{a_n\} \subseteq \mathbb{C}$.

3.15 Theorem CAUCHY CRITERION FOR CONVERGENCE OF A SERIES. *Let $\{a_n\} \subseteq \mathbb{C}$ and*

$$L = \liminf_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right),\ U = \limsup_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right)$$

where $L, U \in \overline{\mathbb{R}}$.

(a) *If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(b) *If $U < 1$, $\sum_{n=1}^{\infty}|a_n| = \infty$ but the series $\sum_{n=1}^{\infty} a_n$ can converge or diverge.*

(c) *If $L = U = 1$, the series $\sum_{n=1}^{\infty}|a_n|$ can converge or diverge, and similarly for $\sum_{n=1}^{\infty} a_n$.*

3.16 Theorem GAUSS CRITERION FOR CONVERGENCE OF A SERIES. *Let $\{a_n\} \subseteq \mathbb{C}$, $\{c_n\} \subseteq \mathbb{R}$ and suppose that*

$$\left|\frac{a_{n+1}}{a_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^p}$$

where $p > 1$ and $|c_n| \leq M < \infty$, $\forall n$.

(a) *If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(b) *If $L \leq 1$, $\sum_{n=1}^{\infty}|a_n| = \infty$ but $\sum_{n=1}^{\infty} a_n$ can converge or diverge.*

3.17 Theorem INTEGRAL CRITERION FOR CONVERGENCE OF A SERIES. *Let $f(x)$, $x \in \mathbb{R}$, a real-valued continuous function, non-negative and non decreasing for $x \geq A$, and let $\{a_n\} \subseteq \mathbb{C}$ a sequence such that $|a_n| = f(n)$ for $n \geq A$. Then*

(a) *$\int_A^{\infty} f(x)dx < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely;*

(b) *$\int_A^{\infty} f(x)dx = \infty \Rightarrow \sum_{n=1}^{\infty}|a_n| = \infty$.*

3.18 Theorem DIRICHLET CRITERION FOR CONVERGENCE OF A SERIES OF PRODUCTS. *Let $\{a_n\} \subseteq \mathbb{C}$ and $\{b_n\} \subseteq \mathbb{R}$ two sequences such that*

(a) *$\left|\sum_{n=1}^{N} a_n\right| \leq M$, for all $N \geq 1$, where $M \geq 0$,*

(b) *$b_{n+1} \leq b_n, \forall n$,*

(c) *$\lim_{n\to\infty} b_n = 0$.*

Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

3.19 Corollary ALTERNATING SERIES CONVERGENCE CRITERION (LEIBNIZ). *Let $\{a_n\} \subseteq \mathbb{R}$ a sequence such that*

(a) $|a_{n+1}| \leq |a_n|, \forall n$,

(b) $a_n = (-1)^{n+1}|a_n|, \forall n$,

(c) $\lim_{n\to\infty} a_n = 0$.

Then the series $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq a_1$.

3.20 Theorem ABEL CRITERION FOR CONVERGENCE OF A SERIES OF PRODUCTS. *Let $\{a_n\} \subseteq \mathbb{C}$ and $\{b_n\} \subseteq \mathbb{R}$ be two sequences such that*

(a) *$\sum_{n=1}^{\infty} a_n$ converges ,*

(b) *b_n is a monotonic bounded sequence.*

Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

3.21 Remark In contrast with most criteria described above, the Abel and Dirichlet criteria do not entail absolute convergence.

3.22 Theorem ABEL-DINI CRITERION FOR CONVERGENCE OF A SERIES OF RATIOS. *If the series $\sum_{n=1}^{\infty} a_n$ diverges such that $S_N = \sum_{n=1}^{N} a_n > 0, \forall N$, and $S_N \underset{N\to\infty}{\longrightarrow} \infty$, then*

(a) *the series $\sum_{n=1}^{\infty} a_n/S_n^{\delta}$ diverges for any $\delta \leq 1$,*

(b) *the series $\sum_{n=1}^{\infty} a_n/S_n^{\delta}$ converges for any $\delta > 1$.*

3.23 Theorem LANDAU CRITERION FOR CONVERGENCE OF A SERIES OF PRODUCTS. *Let $\{a_n\} \subseteq \mathbb{R}$. The series $\sum_{n=1}^{\infty} |a_n|^p$, where $p > 1$, converges $\Leftrightarrow$ the series $\sum_{n=1}^{\infty} a_n b_n$ converges for all sequences $\{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=1}^{\infty} |b_n|^q$ converges, where $q = p/(p-1)$.*

3.24 Remark The Landau theorem implies: if $\sum_{n=1}^{\infty} |a_n|^p < \infty$ and $\sum_{n=1}^{\infty} |b_n|^q < \infty$, where $p > 1$ and $q = p/(p-1)$, then the series $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} |a_n b_n|$ convergent. Further, it gives a necessary condition for the convergence of $\sum_{n=1}^{\infty} |a_n|^p$ *when* $p > 1$.

3.25 Theorem CONVERGENCE OF AN ARITHMETICALLY WEIGHTED MEAN. *Let $\{a_n\} \subseteq \mathbb{C}$. If the series $\sum_{n=1}^{\infty} a_n$ converges, then*

$$\lim_{N\to\infty} \sum_{n=1}^{N} (1 - \frac{n}{N}) a_n = \sum_{n=1}^{\infty} a_n$$

and

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{n}{N} a_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} n\, a_n = 0 .$$

3.26 Theorem CESARO CONVERGENCE. *Let $\{a_n\} \subseteq \mathbb{C}$ a sequence such that $a_n \to a \in \mathbb{C}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = a\ .$$

4. Convergence of transformed series

4.1 Theorem CONVERGENCE OF LINEARLY TRANSFORMED SERIES. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ and $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ be two sequences such that*

$$\sum_{n=0}^{\infty} a_n = A \text{ and } \sum_{n=0}^{\infty} b_n = B$$

where $A, B \in \mathbb{C}$. Then the sequences $\sum_{n=0}^{\infty}(a_n + b_n)$ and $\sum_{n=0}^{\infty} c\, a_n$ converge for any $c \in \mathbb{C}$, and

$$\sum_{n=0}^{\infty}(a_n + b_n) = A + B, \quad \sum_{n=0}^{\infty} c\, a_n = c\ .$$

4.2 Definition CONVOLUTION. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ and $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. We call the sequence*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, n = 0, 1, 2, \ldots$$

the convolution *of the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$. Further, the series $\sum_{n=0}^{\infty} c_n$ is called the* product *of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.*

4.3 Remark We denote the convolution of the sequences a_n and b_n by $a_n * b_n$:

$$a_n * b_n = \sum_{k=0}^{n} a_k b_{n-k}\ .$$

4.4 Theorem SUFFICIENT CONDITION FOR CONVERGENCE OF THE PRODUCT OF TWO SERIES (CAUCHY-MERTENS). . *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ and $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ be two sequences such that*

(a) $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, where $A, B \in \mathbb{C}$, and

(b) $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Then the series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, converges and

$$\sum_{n=0}^{\infty} c_n = AB\ . \tag{Mertens}$$

If, furthermore, $\sum_{n=0}^{\infty} b_n$ converges absolutely, the series $\sum_{n=0}^{\infty} c_n$ converges absolutely.

4.5 Theorem LIMIT OF THE PRODUCT OF TWO SERIES (ABEL). *If $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$, $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ and $\{c_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ are three sequences such that the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge to A, B and C respectively, and if*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} ,$$

then

$$C = AB .$$

4.6 Theorem NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE OF THE PRODUCT OF TWO SERIES. *Let $\{a_n\} \subseteq \mathbb{R}$. The series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$, converges for all sequences $\{b_n\} \subseteq \mathbb{R}$ such that $\sum_{n=0}^{\infty} b_n$ converges $\Leftrightarrow \sum_{n=0}^{\infty} |a_n| < \infty$.*

4.7 Theorem AGGREGATION OF TERMS IN A SERIES. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ be a sequence such that $\sum_{n=0}^{\infty} a_n = A \in \mathbb{C}$, $\{r_n\}_{n=0}^{\infty} \subseteq \mathbb{N}_0$ a monotonically increasing sequence of nonnegative integers such that $r_0 = 0$ and $r_n \to \infty$, and $\{b_n\}_{n=1}^{\infty}$ the sequence defined by*

$$b_n = \sum_{k=r_{n-1}}^{r_n - 1} a_k, n = 1, 2, \dots .$$

Then the series $\sum_{n=1}^{\infty} b_n$ converges and

$$\sum_{n=1}^{\infty} b_n = A .$$

4.8 Definition REARRANGEMENT. *Let $\{k_n\}_{n=0}^{\infty}$ be a sequence of nonnegative integers such that each nonnegative integer appears once and only once in the sequence. If*

$$a'_n = a_{k_n},\ n = 0, 1, 2, \dots ,$$

we call the series $\sum_{n=0}^{\infty} a'_n$ a rearrangement of the series $\sum_{n=0}^{\infty} a_n$.

4.9 Definition COMMUTATIVELY CONVERGENT SERIES. *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ be a sequence such that $\sum_{n=0}^{\infty} a_n$ converges to $A \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} a_n$ is commutatively convergent if all the rearrangements $\sum_{n=0}^{\infty} a'_n$ of the series $\sum_{n=0}^{\infty} a_n$ converge to A.*

4.10 Theorem REARRANGEMENT OF AN ABSOLUTELY CONVERGENT SERIES (DIRICHLET). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $\sum_{n=0}^{\infty} a_n$ converges absolutely to $A \in \mathbb{C}$. Then all the rearrangements of the series $\sum_{n=0}^{\infty} a_n$ converge to A.*

4.11 Theorem REARRANGEMENT OF A CONDITIONNALLY CONVERGENT SERIES (RIEMANN). *Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ be a series such that the series $\sum_{n=0}^{\infty} a_n$ converges conditionally and let*

$$-\infty \le \alpha \le \beta \le \infty .$$

Then there is a rearrangement $\sum_{n=0}^{\infty} a'_n$ *such that*

$$\liminf_{n\to\infty}\left(\sum_{m=0}^{n} a'_m\right) = \alpha \ , \limsup_{n\to\infty}\left(\sum_{m=0}^{n} a'_m\right) = \beta \ .$$

4.12 Theorem EQUIVALENCE BETWEEN ABSOLUTE AND COMMUTATIVE CONVERGENCE. *Let* $\{a_n\} \subseteq \mathbb{C}$ *be a sequence such that the series* $\sum_{n=0}^{\infty} a_n$ *converges. Then* $\sum_{n=0}^{\infty} a_n$ *converges absolutely iff* $\sum_{n=0}^{\infty} a_n$ *converges commutatively.*

4.13 Theorem CONDITION FOR DOUBLE SERIES COMMUTATIVITY. *Let* $\{a_{mn} : m, n = 0, 1, 2, ...\} \subseteq \mathbb{C}$ *be a double sequence such that*

$$\sum_{n=0}^{\infty} |a_{mn}| = b_m, m = 0, 1, 2, \dots$$

and $\sum_{m=0}^{\infty} b_m$ *converges. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}.$$

5. Uniform convergence

5.1 Notation *In this section,* f_n *and* f *refer to functions on a set* E *to the complex numbers* $\mathbb{C}$, i.e. $f_n : E \to \mathbb{C}$ *and* $f : E \to \mathbb{C}$.

5.2 Definition UNIFORM CONVERGENCE. *We say that the sequence of functions* $\{f_n\}_{n=0}^{\infty}$ *converges* uniformly *on* E *to the function* f *if, for any* $\varepsilon > 0$, *there is an integer* N *such that*

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon, \forall x \in E \ .$$

In this case, we write $f_n \to f$ *uniformly on* E.

5.3 Remark We dit that the series $\sum_{i=0}^{\infty} f_i(x)$ converges uniformly on E if the sequence of partial sums $s_n(x) = \sum_{i=0}^{n} f_i(x)$, $n = 0, 1, \dots$ converges uniformly on E.

5.4 Theorem CAUCHY CRITERION FOR UNIFORM CONVERGENCE. *The sequence of functions* $\{f_n\}_{n=0}^{\infty}$ *converges uniformly on* E *to a function* f *if and only if, for any* $\varepsilon > 0$, *there is an integer* N *such that*

$$m, n \geq N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon, \forall x \in E \ .$$

5.5 Theorem SUPREMUM CRITERION FOR UNIFORM CONVERGENCE. *Suppose*

$$\lim_{n\to\infty} f_n(x) = f(x), \forall x \in E \ ,$$

and let

$$M_n = \sup_{x \in E} |f_n(x) - f(x)| \ .$$

Then, $f_n \to f$ *uniformly on* E *if and only if* $\lim_{n\to\infty} M_n = 0$.

5.6 Theorem WEIERSTRASS UNIFORM CONVERGENCE CRITERION. *Suppose* $|f_n(x)| \le M_n$, $\forall x \in E$, $n = 0, 1, 2, \ldots$ *and* $\sum_{n=0}^{\infty} M_n < \infty$. *Then the series* $\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly on* E.

5.7 Theorem UNIFORM CONVERGENCE AND CONTINUITY. *If* $\{f_n\}_{n=0}^{\infty}$ *is a sequence of continuous functions on* E *and if* $f_n \to f$ *uniformly on* E, *then the function* f *is continuous on* E.

5.8 Remark A sequence of continuous functions $\{f_n\}_{n=0}^{\infty}$ *can* converge to a continuous function f without uniform convergence.

5.9 Theorem CONDITIONS OF UNIFORM CONVERGENCE (DINI). *If*

(a) K *is a compact set,*

(b) $\{f_n\}_{n=0}^{\infty}$ *is a sequence of continuous functions on* K,

(c) $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in K$, *where* f *is a continuous function on* K,

(d) $f_n(x) \ge f_{n+1}(x)$, $\forall x \in K$, $n = 0, 1, 2, \ldots.$,

then $f_n \to f$ *uniformly on* K.

5.10 Theorem UNIFORM CONVERGENCE AND DIFFERENTIATION OF FUNCTIONS OF REAL VARIABLES. *Suppose* $[a, b] \subseteq E \subseteq \mathbb{R}$ *and let* $f_n : E \to \mathbb{C}$, *a sequence of differentiable functions on the interval* $[a,b]$ *such that the sequence* $\{f_n(x_0)\}_{n=0}^{\infty}$ *converges for at least one* $x_0 \in [a,b]$. *If the sequence* $\{f_n'\}_{n=0}^{\infty}$ *converges uniformly on* $[a,b]$, *then* $\{f_n\}_{n=0}^{\infty}$ *converges uniformly on* $[a,b]$ *to a differentiable function* f *on* E, *and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \, , \forall x \in [a,b] \ .$$

5.11 Theorem UNIFORM CONVERGENCE AND DIFFERENTIATION OF FUNCTIONS OF COMPLEX VARIABLES. *Let* $E \subseteq \mathbb{C}$ *and* $f_n : E \to \mathbb{C}$, $n = 0, 1, 2, \ldots$, *a sequence of differentiable functions on* E. *If the sequence* $\{f_n\}_{n=0}^{\infty}$ *converges to a function* f *in* E *and* $\{f_n'\}_{n=0}^{\infty}$ *converges uniformly in* E, *then the function* f *is differentiable in* E *and*

$$f'(z) = \lim_{n\to\infty} f_n'(z), \forall z \in E.$$

6. Proofs and references

1. Rudin (1976, Chapters 1 and 3).

1.15. Royden (1968, Section 2.4, pp. 35-38).

1.22 – 1.23 Iyanaga and Kawada (1977, article 374, p. 1162).

2. Rudin (1976, Chapter 3).

2.2. Ahlfors (1979, Chapter 1, p. 62), Gillert, Küstner, Kellwich and Kästner (1986, Section 18.1, p. 422), and Rudin (1976, Chapter 3).

2.6. Rudin (1976, Theorem 3.20, p. 57), and Gillert et al. (1986, Section 18.1, p. 420).

3.1. Rudin (1976, Chapter 3).

3.2. Gillert et al. (1986, Section 18.2, p. 428).

3.3. Knopp (1956, Section 2.6.2, Theorem 1, p. 49, and Section 3.3, Theorem 1, p. 61).

3.15. Devinatz (1968, section 3.2.4, p. 112) and Taylor (1955, Section 17.4, pp. 567-568).

3.16. Taylor (1955, Section 17.4, pp. 568-569).

3.17. Taylor (1955, Section 17.21, pp. 551-553).

3.18. Anderson (1971, Lemma 8.3.1, p. 460).

3.19. Piskounov (1980, 1980, chapitre XVI(7), pp. 294-296).

3.22. Hardy, Littlewood and Polya (1952, Theorem 162, p. 120) and Knopp (1956, Section 2.6.2, Theorem 1, p. 49, and Section 3.3, Theorem 1, p. 61).

3.23. Beckenbach and Bellman (1965, Chapter 3, Theorem 11, pp. 116-117).

3.25. Fuller (1976, Lemma 3.1.5, p. 112).

4. Rudin (1976, Chapter 3).

4.4. Rudin (1976, Section 3.50, pp. 74-75), Taylor (1955, Section 17.6, pp. 575-580) and Iyanaga and Kawada (1977, article 374, p. 1162).

4.6. Beckenbach and Bellman (1965, Chapter 3, Theorem 12, p. 117).

4.7. Gillert et al. (1986, Chapter 18, p. 428).

4.9. Gillert et al. (1986, Section 18.2, p. 429).

4.10 - 4.11. Iyanaga and Kawada (1977, article 374, p. 1162).

4.12. Gillert et al. (1986, section 18.2, p. 429).

5. Ahlfors (1979, Chapter 2) and Rudin (1976, Chapter 7).

5.2. Ahlfors (1979, Section 2.3, p. 36) and Rudin (1976, Definition 7.7, p. 147).

5.4. Ahlfors (1979, Section 2.3, p. 36) and Rudin (1976, Theorem 7.8, p. 147).

5.5. Rudin (1976, Theorem 7.9, p. 148).

5.6. Rudin (1976, Theorem 7.10, p. 148).

5.7. Ahlfors (1979, Section 2.3, p. 36) and Rudin (1976, Theorem 7.12, p. 150).

5.8. Rudin (1976, Sections 7.12 and 7.6, p. 150 and 146).

5.9. Rudin (1976, Theorem 7.13, p. 150).

5.10. Rudin (1976, Theorem 7.17, p. 152).

Other useful references include: Devinatz (1968), Gradshteyn and Ryzhik (1980), Rudin (1987), and Spiegel (1964).

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